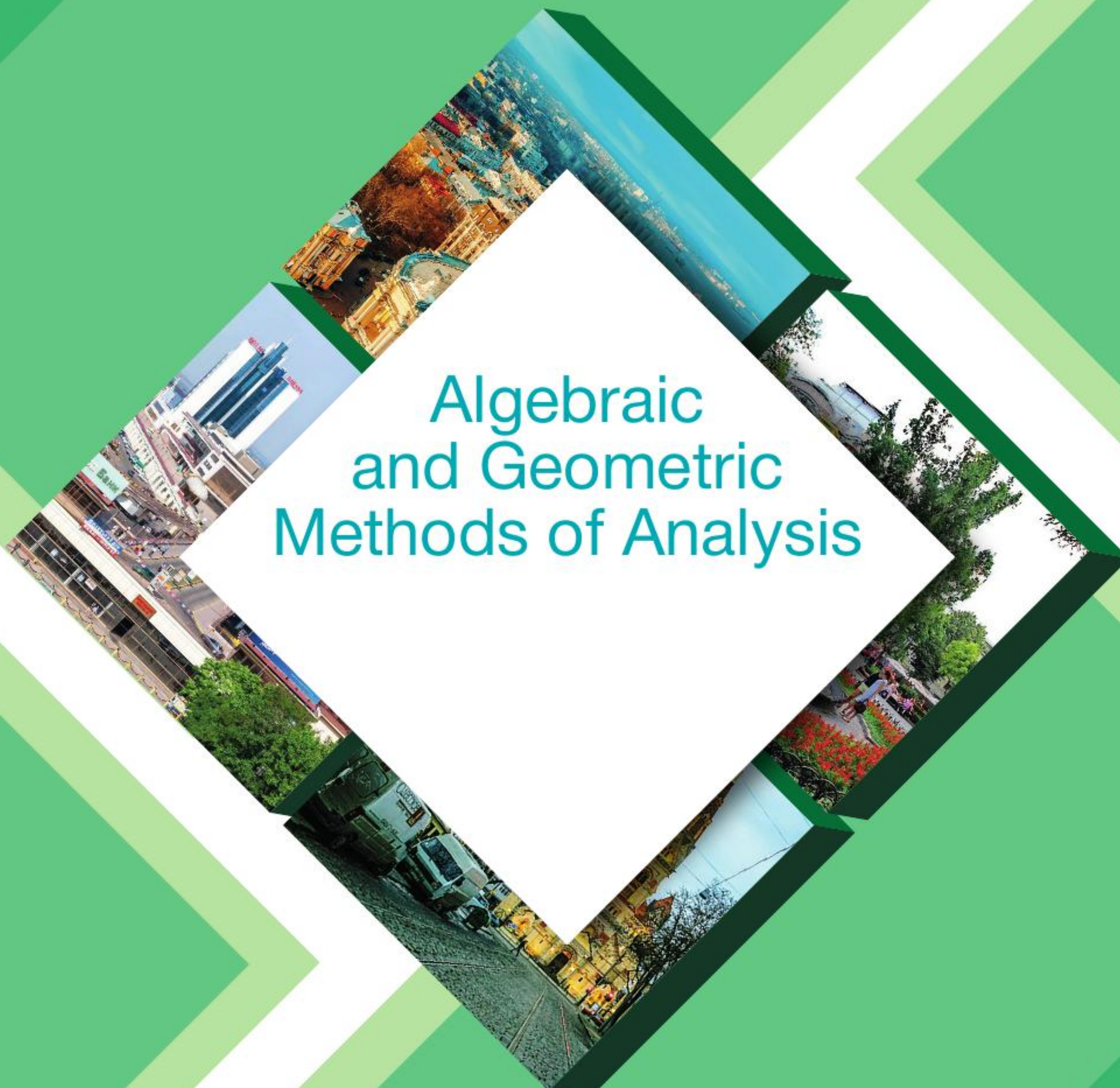
International
Online Conference



Algebraic and Geometric Methods of Analysis

May 26-29, 2025
Odesa, Ukraine

The purpose of this conference is to bring together researchers in geometry, topology, algebra, analysis and dynamical systems and to provide for them a forum to present their recent work to colleagues from different nationalities. This way we aim to stimulate discussion about the latest findings in geometrical and topological methods in analysis and to increase international collaboration.

The conference continues the traditional annual conference «Geometry in Odesa» holding from 2004, and hosted by Odesa National University of Technology (Odesa National Academy of Food Technologies till 2021). From 2017 the conference was renamed to «Algebraic and geometric methods of analysis» (AGMA).

The Conference languages: Ukrainian and English.

List of topics

- Algebraic methods in geometry
- Differential geometry in the large
- Geometry and topology of differentiable manifolds
- General and algebraic topology
- Dynamical systems and their applications
- Geometric and topological methods in natural sciences
- Geometric problems in mathematical analysis

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On the Brody hyperbolicity

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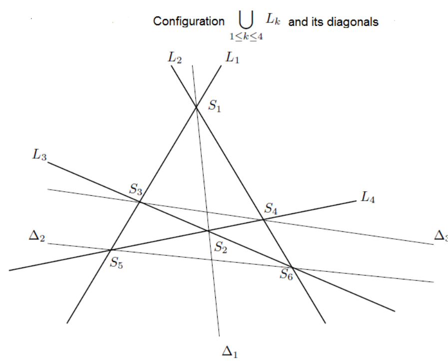
Definition 1. Let $H_1, ..., H_m$, $m \geqslant 2n$, be a configuration of 2n hyperplanes in general position of $\mathbb{C}P^n$. We call diagonal, the line passing through the two points $\cap_{i\in I} H_i$ and $\cap_{j\in J} H_j$, where $|I| = |J| = n$ and $I \cap J = \emptyset$. Here $|I|$ denotes the cardinal of I.

Theorem 2. *Let $H_1, ..., H_{2n}$ be $(2n)$ projective hyperplanes in general position in $\mathbb{C}P^n$. Then there are $\frac{1}{2}C_{2n}^n$ diagonals $\Delta_1, ..., \Delta_{\frac{1}{2}C_{2n}^n}$ such that for any non constant holomorphic curve $f : \mathbb{C} \longrightarrow \mathbb{C}P^n \setminus \cup_{i=1}^{2n} H_i$, there exists $k_f \in \{1, ..., \frac{1}{2}C_{2n}^n\}$ such that $f(\mathbb{C}) \subset \Delta_{k_f}$.*

Corollary 3 (This is how we prove the Green Theorem)**.** *Any holomorphic curve that lies in the complement of $2n + 1$ hyperplanes in general position in $\mathbb{C}P^n$, is constant.*

Theorem 4. *(E. Borel) Let $H = \cup_{i=1}^4 H_i$ a collection of complex projective lines in general position in $\mathbb{C}P^2$. Then any non constant map $f : \mathbb{C} \to \mathbb{C}P^2 \setminus H$, lies in one of the diagonales $(\Delta_i)_{i=1,2,3}$. Where Δ_i are the projective lines passing each through a double points of H.*



Theorem 5. *Let L_1, L_2, L_3, L_4 and L_5 complex hyperplanes in general position in $\mathbb{C}^3$, then for every holomorphic curve $G : \mathbb{C} \to \mathbb{C}^3$ such that $G(\mathbb{C}) \cap (\cup_{i=1}^5 L_i) = \emptyset$, there exists a complex line L in $\mathbb{C}^3$ such that $G(\mathbb{C}) \subset L$. Moreover, the complementary of five complex lines in $\mathbb{C}^3$ is not Brody hyperbolic. (This result is also true in higher dimension)*

Remark: The projection of G into the complex projective space $\mathbb{C}P^2$ is constant.

Definition 6. For $n \geqslant 3$ and $\mathcal{L} = (L_1, ..., L_n)$ a family of real subspaces of $\mathbb{R}^6$ of real codimension 2. Then we say that $\mathcal{L}$ is in general position if for every 3-tuple (i, j, l) of distinct integers $i, j, l \in \{1, ..., n\}$,

$$Span_{\mathbb{R}}(L_i^{\perp}, L_j^{\perp}, L_l^{\perp}) = \mathbb{R}^6.$$

We note that if L is a real subspace in $\mathbb{R}^6$, then $L^{\perp}$ denotes the orthogonal complement of L.

Theorem 7. *Let L_1, L_2, L_3, L_4 be four complex lines in $\mathbb{C}^3$. Then there exists a real subspace L of $\mathbb{R}^6$, of real dimension four, such that (L, L_i, L_j) are in general position for all $j \neq i$, $j, i \in \{1, ..., 4\}$, and there exists a non constant holomorphic curve $g : \mathbb{C} \to \mathbb{C}^3$, such that*

$$g(\mathbb{C}) \bigcap (\bigcup_{i=1}^{4} L_i \bigcup L) = \emptyset$$

ie, the complementary of this configuration in $\mathbb{C}^3$ is not Brody hyperbolic.
Remark: The projection of G into the complex projective space $\mathbb{C}P^2$ is not constant.

Here π denotes the canonical projection from $\mathbb{C}^3 \setminus \{0\}$ into $\mathbb{C}P^2$ and $\pi(g) := \pi \circ g$. Notice that $\pi(g)$ is well-defined since $g(\mathbb{C}) \subset \mathbb{C}^3 \setminus \{0\}$.

Theorem 8. *The complementary of five real subspaces $\tilde{L}_i$, $i = 1 \dots 5$ of real dimension 5 in $\mathbb{C}^3$ is Brody hyperbolic. That is to say that any holomorphic map $g : \mathbb{C} \to \mathbb{C}^3 \setminus \cup_{i=1}^{5} \tilde{L}_i$ is constant.*

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On the mapping of surfaces of Euclidean spaces

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Let us consider the Euclidean spaces E_4 and $\overline{E}_4$ as completely orthogonal subspaces in the proper Euclidean space E_8, having one common point O. Let V_2 and $\overline{V}_2$ be smooth surfaces in E_4 and $\overline{E}_4$ respectively.

We will study differentiable one-to-one mapping T of a domain $\Omega \subset V_2$ onto a domain $\overline{\Omega} \subset \overline{V}_2$. If a point X_1 inscribes a domain Ω, and $X_2 = T(X_1) \subset \overline{\Omega}$, then a point X with radius vector $\overrightarrow{X} = \overrightarrow{X}_1 + \overrightarrow{X}_2$ inscribes a certain two-dimensional surface V_2^*, called the graph of the mapping T [1].
In [2], [3], [4], it is shown that in this case, each surface V_2 and $\overline{V}_2$, there arise orthogonal sets $\delta_2 \subset V_2$ and $\overline{\delta}_2 \subset \overline{V}_2$.

The following theorems proved

Theorem 1. *The sets δ_2 and $\overline{\delta}_2$ correspond to the mapping T if and only if one of the following conditions is satisfied:*
1) *the sets δ_2 and $\overline{\delta}_2$ coincide with the base of the mapping T,*
2) *the mapping T is conformal.*

Theorem 2. *If the surfaces V_2 and $\overline{V}_2$ carry conjugate sets and these sets correspond, then the sets δ_2 and $\overline{\delta}_2$ serve as the basis of the mapping T if and only if the condition.*

$$\overrightarrow{C}_{12}\left[\left(C_{12}^{4}\overline{\gamma}^{1i} - C_{12}^{3}\overline{\gamma}^{2i}\right)\overrightarrow{e}_{4+i}\right] = 0$$

is satisfied.

Theorem 3. *The base of the mapping T harmonically separates the conjugate sets Σ_2 and $\overline{\Sigma}_2$ if and only if condition $\overrightarrow{C}_{12}\,(C^3_{12}\overrightarrow{e}_1 - C^4_{12}\overrightarrow{e}_2) = 0$ is satisfied.*

Theorem 4. *A pair of surfaces V_2, $\overline{V}_2$, carrying conjugate sets corresponding to the mapping T is determined by specifying four functions of two arguments.*
Note that an arbitrary pair of surfaces V_2, $\bar{V}_2$, is defined by specifying six functions of two arguments (two functions for each of the surfaces $V_2 \subset E_4$ and $\bar{V}_2 \subset E_4$- and two functions for specifying the mapping $T : \Omega \to \bar{\Omega}$).

Theorem 5. *If the surfaces V_2 and $\overline{V}_2$ carry orthogonal conjugate networks and these networks correspond, then the networks δ_2 and $\overline{\delta}_2$ correspond in this mapping T if and only if one of the following conditions is satisfied:*

1) *$C^3_{12} = 0, C^4_{12} \neq 0$ (or $C^4_{12} = 0, C^3_{12} \neq 0$). Here C^3_{12}, C^4_{12} do not vanish simultaneously, since rang $\left\|C^n_{ij}\right\| = 3$. Geometrically, this means that the vector $\overrightarrow{C}_{12}$ is either collinear with the vector $\overrightarrow{\mathcal{E}}_3$, or with $\overrightarrow{\mathcal{E}}_4$.*

2) *The mapping T is conformal. Considering that the vector $\overrightarrow{C}_{12}$ is the following decomposition.*

$$\overrightarrow{C}_{12} = \overrightarrow{m} - \overrightarrow{\overline{m}}$$

we have

Corollary 6. *Let $\overline{\Sigma}_2 = T\,(\Sigma_2)$ and let the sets Σ_2 and $\overline{\Sigma}_2$ be orthogonal and conjugate. The sets Σ^*_2 of the graph V^*_2 is a set of curvature lines with respect to the mean normal if and only if*

$$\overrightarrow{\mu}^* \cdot \overrightarrow{m} = \overrightarrow{\mu}^* \cdot \overrightarrow{\overline{m}}$$

where $\overrightarrow{\mu}^*$ is the mean normal vector of the surface V^*_2.

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Deformations, fundamental groups, and Zariski pairs in classification of algebraic curves and surfaces

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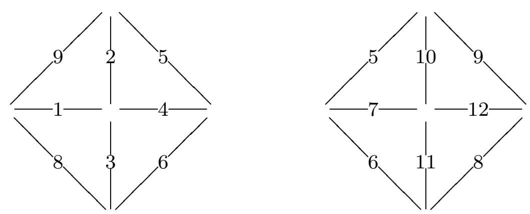
Classification of algebraic surfaces and curves has been a major mathematical problem over the years. In the talk I will focus on both studies. The background and some new results will be presented, especially by using topological and algebraic methods in geometry.

Classification of algebraic surfaces. : Studying how algebraic surfaces change into unions of planes is a fascinating area of research. These changes, or deformations, help to understand the geometry and topology of these surfaces, especially by looking at singular points and invariants like fundamental groups. Planar and non-planar deformations of algebraic surfaces involve breaking down these surfaces into unions of planes. Non-planar deformations are more complex than the planar ones because they involve the connection of edges to form high multiple singularities. Both types are crucial for understanding the geometry and topology of these surfaces, with applications in algebraic geometry, topology, and physics.

In the following figure we can see one example of a non-planar deformation, one of many that are of great interest to mathematicians in topology and algebraic geometry. If we glue the two pieces in the figure along their external edges, we will get a non-planar deformation, with complicated singularities along this gluing.



We can compute the fundamental group G of the complement of the branch curve of an algebraic surface. The fundamental group is an invariant of the surface. Via the deformation, we can derive the dual graph that represents the group and contributes a lot of information to the classification.

If group G is complicated, we can compute the fundamental group of the Galois cover of the surface, and it is an invariant of the surface as well.

We give new results in this area of research; selected references are [1]-[6].

Classification of algebraic curves. : We classify algebraic curves using Zariski pairs, which are pairs of curves that have the same combinatorial structure but differ in their topological properties. By studying these pairs, we gain insight into the unique characteristics of each curve.

Our research focuses on line arrangements and conic-line arrangements. The deformations of these arrangements are interesting objects by themselves, and the study helps us to see how different changes in the curves affect the fundamental groups related to them. The computations give us a better understanding of their underlying topology.

Concerning line arrangements: Zariski pairs of line arrangements cannot be in the same component of the moduli space. This is a powerful tool to study the moduli space of line arrangements.

Concerning conic-line arrangements: there are no Zariski pairs of degree ≤ 5. For degree 7, there are already some interesting examples of Zariski pairs.

We will see the correspondence between curves and fundamental groups and understand the rules given by Zariski pairs.

In the following figure we give an example of a Zariski pair of degree 8. Such an example will be explained later, among other examples from [7]-[10].

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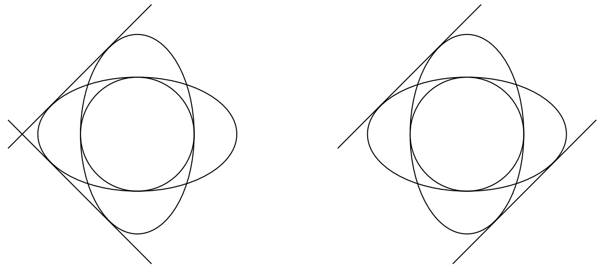

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Homogeneous homoderivations on graded associative rings

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As a fact, the concept of homogeneous derivations was introduced by Kanunnikov (2018)[1]. Let Λ be a G-graded ring. An additive mapping $\kappa\colon \Lambda \to \Lambda$ is called homogeneous derivation if:

- $\kappa(xy) = \kappa(x)y + x\kappa(y)$ for all $x, y \in \Lambda$.
- $\kappa(r) \in H(\Lambda)$ for all $r \in H(\Lambda)$.

While the year 2000 came, a classical definition concerning of homoderivation was delivered via article [2], where an additive mapping is a homoderivation concerning Λ like from Λ to Λ, where Λ

is a ring. In other expressions, (from a ring to itself) satisfy $\kappa(xy) = \kappa(x)\kappa(y)+\kappa(x)y+x\kappa(y)$ where x and y in Λ. Every homogeneous derivation is a derivation. However, the converse statement is not true, as there exist derivations that are not homogeneous.

Over the last 70 years, researchers have been interested in understanding the structure and commutativity of ring R using specific types of mappings called derivations. Various authors have widely studied this topic. Like [3]. In 1957, the study of commutativity of prime rings with derivations was initiated. Since then, the relationship between the commutativity of rings and the existence of specific types of derivations has attracted many researchers. The main result in this context is that a prime ring R with a nonzero centralizing derivation d must be a commutative ring. Graded rings have various applications in geometry and physics, and appear in various contexts, from elementary to advanced levels. Based on the rich heritage of ring theory, many researchers have attempted to extend and generalize various classical results to graded settings.

In this paper, Λ represents an associative ring with the center $Z(\Lambda)$, and G is an abelian group with identity element e. For $x, y \in \Lambda$, we write $[x, y]$ for Lie product $xy - yx$ and for a nonempty subset S for Λ, we write $C_\Lambda(S) = \{x \in \Lambda \mid [x, S] = 0\}$ for the centralizer of S in Λ. A ring Λ is G-graded if there is a family $\{\Lambda_g, g \in G\}$ of additive subgroups Λ_g of $(\Lambda, +)$ such that $\Lambda = \bigoplus_{g\in G} \in \Lambda_g$ and $\Lambda_g\Lambda_h \subseteq \Lambda_{gh}$ for every $g, h \in G$. The additive subgroup Λ_g called the homogeneous component of Λ. The set $H(\Lambda) = \cup_{g\in G}\Lambda_g$ is the set of homogeneous elements of Λ.

Let η be a right (resp. left) ideal of a graded ring Λ. Then η is said to be a graded right (resp. left) ideal if $\eta = \bigoplus_{g\in G} \eta_G$, where $\eta_g = (\eta \cap \Lambda_g)$ for all $g \in G$. That is, for $x \in \eta, x = \sum_{g\in G} x_g$, where $x_G \in \eta$ for all $g \in G$. A graded ring Λ is said to be gr-prime (gr-semiprime) if $a\Lambda b = \{0\}$ implies $a = 0$ or $b = 0$ (if $a\Lambda a = \{0\}$ then $a = 0$), where $a, b \in H(\Lambda)$. Moreover, a graded ring Λ is a gr-semiprime ring if the intersection of all the gr-prime ideals is zero.

Here, we establish interesting results related to homogeneous homoderivations. We prove the existence of a non-trivial family of homoderivations that are not homogeneous on graded rings. Furthermore, based on homogeneous homoderivations, we extend certain existing significant results in the context of prime (resp. semiprime) rings to gr-prime (resp. gr-semiprime) rings.

Theorem 1. *Let Λ be a gr-semiprime ring with a 2-torsion free property. If κ is a homogeneous homoderivation and $c \in H(\Lambda)$ such that $[c, \kappa(x)] \in Z(\Lambda)$ for all $x \in \Lambda$, then $\kappa = 0$ or $c \in Z(\Lambda)$.*

Theorem 2. *Let Λ be a 2-torsion free gr-semiprime ring with a 2-torsion free property and gr-prime ideal η be a gr-prime ideal Λ. Suppose κ_1 and κ_2 be homoderivations of Λ. Suppose that at least one of κ_1 and κ_2 is homogeneous and their composition $\kappa_1\kappa_2$ is a derivation. Then either $\kappa_1 \in \eta$ or $\kappa_2 \in \eta$.*

Proposition 3. *Let Λ be a gr-prime ring and η a non zero graded left ideal of Λ. If κ is a non zero homogeneous homoderivation of Λ, then its restriction on η is non zero.*

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Thomae formulas in application to finding reality conditions for integrable hierarchies

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Spectral curves of the KdV, sine-Gordon, and mKdV hierarchies all belong to the family of hyperelliptic curves of the form

$$\mathcal{C}: \qquad f(x,y) \equiv -y^2 + \mathcal{P}(x) \equiv -y^2 + x^{2g+1} + \textstyle\sum_{i=1}^{2g} \lambda_{2i+2} x^{2g-i} = 0, \tag{1}$$

where the genus g of $\mathcal{C}$ coincides with the number of gaps of a hamiltonian systems in the hierarchy. Parameters $\lambda = \{\lambda_{2i+2}\}_{i=1}^{2g}$ serve as integrals of motion.

Let $\mathrm{Jac}(\mathcal{C}) = \mathbb{C}^g/\{\omega,\omega'\}$ be the Jacobian variety of $\mathcal{C}$ with respect to the lattice generated by columns of not normalized period matrices $\omega = (\omega_{i,j})$, $\omega' = (\omega'_{i,j})$ defined by

$$\omega_{i,j} = \int_{\mathfrak{a}_j} \mathrm{d}u_{2i-1}, \qquad \omega'_{i,j} = \int_{\mathfrak{b}_j} \mathrm{d}u_{2i-1}, \quad \text{where} \quad \mathrm{d}u_{2i-1} = \frac{x^{g-i}\mathrm{d}x}{-2y}.$$

Second kind period matrices $\eta = (\eta_{i,j})$, $\eta' = (\eta'_{i,j})$ are obtained from the second kind differentials associated with the first kind differentials $\mathrm{d}u = (\mathrm{d}u_1, \mathrm{d}u_3, \ldots, \mathrm{d}u_{2g-1})^t$ defined above.

Each curve $\mathcal{C}$ is uniformized by means of the multiply periodic $\wp$-functions

$$\wp_{i,j}(u) = -\frac{\partial^2 \log \sigma(u)}{\partial u_i \partial u_j}, \qquad \wp_{i,j,k}(u) = -\frac{\partial^3 \log \sigma(u)}{\partial u_i \partial u_j \partial u_k},$$

which generalize the Weierstrass $\wp$-function to higher genera. The sigma function is defined by

$$\sigma(u) = C \exp\big(-\tfrac{1}{2} u^t \varkappa u\big) \theta[K](\omega^{-1}u; \omega^{-1}\omega'), \tag{2}$$

see [1, Eq.(2.3)], where $[K]$ is the characteristic of the vector K of Riemann constants, and $\varkappa = \eta\omega^{-1}$.

The mentioned completely integrable equations have the following finite-gap solutions, $b \in \mathbb{R}$, $c_i \in \mathbb{R}$,

KdV	$w_t = 6ww_x + w_{xxx}$	$w(x,t) = -b\wp_{1,1}(u), \quad u = -b(x,t,c_5,\ldots,c_{2g-1})^t + \omega K,$
sine-Gordon	$\phi_{t,x} = 4\sin\phi$	$\phi(x,t) = \imath \log\big(-\lambda_{4g}^{-1/2}\wp_{1,2g-1}(u)\big), \; u = \imath b(x,c_3,\ldots,c_{2g-3},t)^t + \omega K,$
sinh-Gordon	$\phi_{t,x} = -4\sinh\phi$	$\phi(x,t) = \log\big(-\lambda_{4g}^{-1/2}\wp_{1,2g-1}(u)\big), \; u = b(x,c_3,\ldots,c_{2g-3},t)^t + \omega K,$
mKdV	$w_t = 6\varsigma w^2 w_x - w_{xxx}$	$w(x,t) = -\dfrac{\mathrm{b}\wp_{1,1,2N-1}(u)}{2\wp_{1,2N-1}(u)}, \quad u = \mathrm{b}(x,-4\mathrm{b}^2 t, c_5,\ldots,c_{2g-1})^t + \omega K.$

In the case of defocusing mKdV, assign $\varsigma = 1$, $\mathrm{b} = b$. In the case of focusing mKdV, $\varsigma = -1$, $\mathrm{b} = \imath b$.

The reality conditions require all solutions to be real-valued and bounded functions of real variables x, and t. That is, the reality conditions are specified by the choice of a path in $\mathrm{Jac}(\mathcal{C})$ where a solution of the system in question is real-valued. An answer to this question is obtained from the analysis of values of the σ-function at half-periods on the spectral curve $\mathcal{C}$.

Recall, that after separation of variables, a g-gap hamiltonian system in one of the mentioned hierarchies splits into g one-particle systems, whose phase trajectories are determined by (1), namely by $f(x_i,y_i) = 0$, $i = 1, \ldots, g$, where x_i serves as the coordinate, and y_i as the momentum. Thus, $-\mathcal{P}(x)$ serves as the potential, and so roots of $\mathcal{P}$ serve as turn points. Therefore, the Abel image of a divisor composed from g points, one on each one-particle trajectory, goes through half-periods. For the purpose of a bounded solution, these half-periods should be non-singular.

The Thomae formulas introduce a connection between null values of the theta function with characteristics (or its first non-vanishing derivative), called theta constants (or theta derivatives), on the one hand, and x-coordinates of the branch points which produce half-periods corresponding to the characteristics, on the other hand. Instead of theta constants and theta derivatives we use values of the σ-function (or its first non-vanishing derivative) at half-periods. Each half-period is represented by a partition on the set of indices of branch points.

Let $\mathcal{S} = \{0, 1, 2, \ldots, 2g+1\}$ be the set of indices of all branch points, and 0 stands for infinity. A partition $\mathcal{I}_0 \cup \mathcal{J}_0 = \mathcal{S}$, $\mathrm{card}\mathcal{I}_0 = \mathrm{card}\mathcal{J}_0 = g+1$, represents a characteristic of multiplicity 0, or an even non-singular characteristic, which describes a half-period Ω_0 such that $\sigma(\Omega_0) \neq 0$. A partition $\mathcal{I}_\mathfrak{m} \cup \mathcal{J}_\mathfrak{m} = \mathcal{S}$, $\mathrm{card}\mathcal{I}_\mathfrak{m} = g+1-2\mathfrak{m}$, $\mathrm{card}\mathcal{J}_\mathfrak{m} = g+1+2\mathfrak{m}$, represents a characteristic of multiplicity $\mathfrak{m}$, which describes a half-period $\Omega_\mathfrak{m}$ such that $\partial_{u_1}^{\mathfrak{m}}\sigma(\Omega_\mathfrak{m}) \neq 0$ and $\partial_{u_1}^{\mathfrak{r}}\sigma(\Omega_\mathfrak{m}) = 0$ if $0 \leqslant \mathfrak{r} < \mathfrak{m}$. All half-periods represented by partitions $\mathcal{I}_\mathfrak{m} \cup \mathcal{J}_\mathfrak{m} = \mathcal{S}$ with $\mathfrak{m} > 0$ are called singular, due to $\wp$-functions have singularities at such half-periods.

In the fundamental domain of $\mathrm{Jac}(\mathcal{C})$, there exist 2^{2g} half-periods. In the case of a real curve (with real parameters λ), these half-periods form 2^g lines parallel to the real axes, and 2^g lines parallel to the imaginary axes, each line contains 2^g half-periods. It is proven, see [2, Propositions 2, 3], [3, Theorem 4], that there exists only one line parallel to the real axes, and only one line parallel to the imaginary axes, which contains no singular half-periods. Any of the two lines can serve as the domain for finite-gap solutions of the integrable systems.

Further, the reality conditions require real values of solutions, that is for $s \in \mathbb{R}^g$

KdV	$\wp_{1,1}(s + \omega K) \in \mathbb{R}$,
sine-Gordon	$\lvert\wp_{1,2g-1}(\imath s + \omega K)\rvert^2 = \lambda_{4g}$,
sinh-Gordon	$\lvert\wp_{1,2g-1}(s + \omega K)\rvert^2 = \lambda_{4g}$,
defocusing mKdV	$\dfrac{\wp_{1,1,2N-1}(s + \omega K)}{\wp_{1,2N-1}(s + \omega K)} \in \mathbb{R}$,
focusing mKdV	$\imath\dfrac{\wp_{1,1,2N-1}(\imath s + \omega K)}{\wp_{1,2N-1}(\imath s + \omega K)} \in \mathbb{R}$.

Direct computations of the above expressions show that not all real curves can serve as spectral curves of the mentioned integrable hierarchies. As proven in [2, Propositions 4, 5], [3, Theorem 5], [4, Theorem 5], the requires reality conditions are satisfied on the following curves.

Theorem 1. *Hyperelliptic curves which possess a branch point at infinity, and all other branch points are real, serve as spectral curves for the KdV hierarchy.*

Theorem 2. *There exist two types of real hyperelliptic curves which satisfy the reality conditions for the sine(sinh)-Gordon equation and the mKdV equation:*

(RC) hyperelliptic curves which possess a branch point at infinity, a branch point at the origin, and all other branch points are real.

(IC) hyperelliptic curves which possess a branch point at infinity, a branch point at the origin, and all other branch points split in complex conjugate pairs.

Curves (RC) serve as spectral for the sinh-Gordon, and defocusing mKdV hierarchies. Curves (IC) serve as spectral for the sine-Gordon, and focusing mKdV hierarchies.

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On the derivations and automorphisms of Clifford algebras over countable-dimensional vector spaces

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Let $\mathcal{C}\ell(V,f)$ denote the Clifford algebra of a vector space V over a field $\mathbb{F}$ of characteristic not equal to 2, generated by V with unit 1 and defining relations $v^2 = f(v)\cdot 1$, where f is a nondegenerate quadratic form; see [4, 5].

Assume the ground field $\mathbb{F}$ is algebraically closed. According to N. Jacobson [2], if the dimension of the vector space V is even, then the Clifford algebra $\mathcal{C}\ell(V,f)$ is isomorphic to a matrix algebra; if the dimension of V is odd, then $\mathcal{C}\ell(V,f)$ is isomorphic to the direct sum of matrix algebras. For an infinite dimensional vector space V, the Clifford algebra $Cl(V,f)$ is a locally matrix algebra; see [1].

Two main families of derivations and automorphisms of Clifford algebras are known:

(1) Inner derivations and inner automorphisms.
(2) Bogolyubov derivations and Bogolyubov automorphisms.

The Clifford algebra $\mathcal{C}\ell(V,f)$ is graded by the cyclic group of order 2, expressed as

$$\mathcal{C}\ell(V,f) = \mathcal{C}\ell(V,f)_{\bar{0}} + \mathcal{C}\ell(V,f)_{\bar{1}}.$$

A derivation D of the algebra $\mathcal{C}\ell(V,f)$ is called *even* if

$$D(\mathcal{C}\ell(V,f)_{\bar{0}}) \subseteq \mathcal{C}\ell(V,f)_{\bar{0}}, \qquad D(\mathcal{C}\ell(V,f)_{\bar{1}}) \subseteq \mathcal{C}\ell(V,f)_{\bar{1}};$$

and *odd* if

$$D(\mathcal{C}\ell(V,f)_{\bar{0}}) \subseteq \mathcal{C}\ell(V,f)_{\bar{1}}, \qquad D(\mathcal{C}\ell(V,f)_{\bar{1}}) \subseteq \mathcal{C}\ell(V,f)_{\bar{0}}.$$

We describe derivations of the Clifford algebra associated with a nondegenerate quadratic form on a countable-dimensional vector space over an algebraically closed field of characteristic not equal to 2. Any nonzero derivation D of $\mathcal{C}\ell(V,f)$ can be uniquely represented as a sum: $D = \sum_S \alpha_S \,\mathrm{ad}(v_S)$, where $0 \neq \alpha_S \in \mathbb{F}$.

- For an *even* derivation D, the subsets S are finite, nonempty subsets of $\mathbb{N}$ of even order, and each $i \in \mathbb{N}$ belongs to at most finitely many subsets S.
- For an *odd* derivation D, the subsets S are finite subsets of $\mathbb{N}$ of odd order, and each $i \in \mathbb{N}$ lies in all but finitely many subsets S.

Additionally, we characterize when a nonzero even derivation of the Clifford algebra is a Bogolyubov derivation and when a Bogolyubov derivation corresponding to a skew-symmetric linear transformation is an inner derivation.

Now, suppose the field $\mathbb{F} = \mathbb{R}$ is the field of real numbers, and let $f : V \to \mathbb{R}$ be a positive definite quadratic form. In this case, the Clifford algebra $\mathcal{C}\ell(V,f)$ naturally inherits the structure of a normed algebra. In a 2022 MathOverflow discussion, M. Ludewig (see [3]) posed the question of whether every automorphism of $\mathcal{C}\ell(V,f)$ is continuous with respect to this norm.

In response, we construct an algebraic automorphism of $\mathcal{C}\ell(V,f)$ that is not continuous with respect to the given norm.

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Dual Thurtson norm of Euler classes of foliations on closed 3-Manifolds

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In this work we give an upper bound estimate on the dual Thurston norm of the Euler class of an arbitrary smooth foliation $\mathcal{F}$ of dimension one defined on a closed three-dimensional orientable irreducible atoroidal Riemannian manifold M^3.

We present the following result.

Theorem 1. *Let (M^3, g) be a closed oriented three-dimensional irreducible atoroidal Riemannian manifold equipped by a two-dimensional transversely oriented foliation $\mathcal{F}$, whose leaves have the modulus of a mean curvature H bounded above by the constant $H_0 \geq 0$, and M^3 satisfies the following conditions:*

(1) $Vol(M^3) \leq V_0$;
(2) $k_0 \leq K \leq K_0$;
(3) $inj(M^3) \geq i_0$.
(4) $stsys_1(M^3) \geq s_0$

for some fixed constants $V_0 > 0, i_0 > 0, k_0 < K_0, s_0 > 0$, bounding the volume $Vol(M^3)$, the sectional curvature K of M^3, the injectivity radius $inj(M^3)$ and the 1-dimensional stable systole $stsys_1(M^3)$.

Then there exists the conctant $C(H_0, V_0, i_0, k_0, K_0, s_0)$ such that the dual Thurston norm $||e(T\mathcal{F})||^*_{Th}$ of the Euler class $e(T\mathcal{F})$ of the tangent to $\mathcal{F}$ distribution $T\mathcal{F}$ satisfies the following:

$$||e(T\mathcal{F})||^*_{Th} \leq C.$$

Seeding optimization in the batch crystallization of CAM

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The citric acid monohydrate (CAM) is an important organic substance but, until 1997, the scientific literature covered mostly the kinetics of nucleation [3] and the crystal growth [4] rather than its production via the crystallization by cooling in a stirred tank reactor (STR). The Department of Chemical Engineering at the University "La Sapienza" of Rome decided to fill that sci-tech gap through a meticulous investigation, with three STRs at the laboratories of San Pietro in Vincoli's district (DICMA), on the crystallization in discontinuous (batch) of CAM from aqueous solutions. The author participated in that innovative experience, as experimenter and coder under the supervision of Prof. Barbara Mazzarotta, in the years 1997-1998 [1]. Our specific tasks were to spot the main operating conditions, to modify them until an *optimal* crystal size distribution (CSD), i.e., large-sized homogeneous crystals of CAM, and to write a QBasic program predicting the outcomes of any test in batch reactors [2]. Here we focus on the influence of the *seeding*, i.e., the role played by the CAM seed crystals in the process thanks to their varied sizes and dipping temperatures. All the data, collected and simulated, show that the *light* seed performs better than the heavy seed and that a *low* seeding temperature gives the best CSD. The homogenous distribution of large crystals from a low temperature round-bottomed tank, seeded with small CAM crystals, is due to the maximum efficacy of the driving force provided by the related supersaturation.

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Singularities, Torsion, Cauchy Integrals and their Spectra on Space-Time, III

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The field sources can beidentified as fields ϕ_{AB} , which in a complex Riemannian manifold that models the space-time including field sources, can be re-interpreted as poles or singularities of said manifold such that their integrals can calculate their value through the Cauchy type integrals as the Conway integrals to any loop generated in the local causal structure (light cones) of the space-time around of these fields. These integrals are solutions of the spinor equation associated to the corresponding twistor field equation. A theorem is mentioned on the evidence of field torsion as field invariant and geometrical invariant in poles of Cauchy type integrals in spinor-twistor frame. Then the torsion existence in the space-time induces gravitational waves in a projective bundle. Sources are evidence at least locally, of torsion existence. Therefore exists curvature here. Some conjectures and technical lemmas are mentioned as references of other works which gives enter to a new application conjecture to the respect.

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From Maxwell's equations to relativistic Schrödinger equation via Schwartz Linear Algebra and Killing vector fields on the 2-sphere

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In this work, we develop a comprehensive mathematical framework unifying scalar relativistic quantum mechanics with classical electromagnetic field theory by means of Schwartz-linear algebra. Building upon the foundations introduced by David Carfì in [1, 2, 3, 4], we construct a partial embedding of tempered scalar distributions into spaces of tempered vector-valued fields that carry natural Maxwellian structure.

The key object of our study is an embedding operator $J_{(\eta,f)}$, Schwartz-linear and continuous, that maps a large class of complex wave distributions

$$\psi \in \mathcal{S}'(\mathbb{M}^4, \mathbb{C})$$

into transverse vector fields

$$F \in W = \mathcal{S}'(\mathbb{M}^4, \mathbb{C}^3)$$

via spectral synthesis with polarization. Specifically, the embedding is constructed using a transverse, right-handed polarization frame

$$f : k \mapsto f(k) = (r(k), s(k)) \in \mathbb{R}^3 \times \mathbb{R}^3,$$

defined on the dual space $\mathbb{M}_4^*$ minus Π, where Π is a singular plane and r is a killing vector field on the 2-sphere extended omogeneously to the whole dual of Minkovski space-time minus Π. The Maxwell's basis

$$w : k \mapsto w_k := \eta_k(r(k) + is(k)),$$

with $\eta_k(x) = e^{i\langle k,x\rangle}$, forms a Schwartz linearly independent system of circularly polarized plane waves, generating a vast subspace S of Schwartz-Maxwell electromagnetic field space W. The map

$$J_{(\eta,f)} : \psi \mapsto J_{(\eta,f)}(\psi) = \int_{\mathbb{M}_4^*} (\psi)_\eta w$$

embeds scalar wave distributions into the Maxwell-Schwartz field space, provided that the complex wave distribution ψ admits a momentum representation $(\psi)_\eta$ vanishing around the singular plane Π.

We show that this embedding preserves eigenstructures of quantum observables diagonale on η. The momentum operator $\hat{p} = -i\hbar\nabla$ and energy operator $\hat{E} = i\hbar\partial_0$ act compatibly through $J_{(\eta,f)}$, and w_k are simultaneous eigenfunctions of $\hat{p}$ and curl, with eigenvalues $\hbar\vec{k}$ and $|\vec{k}|$, respectively. The operator $\hbar$ curl is therefore identified with the momentum magnitude operator on the subspace S of the Maxwell-Schwartz space.

Furthermore, the theory is extended to general embeddings $J_{(\beta,f)}$, constructed from arbitrary Schwartz bases β and smooth frame fields $f : D \to \mathbb{C}^3$ with the same domain of β. These embeddings commute with all observables diagonal in the basis β, yielding a functorial structure.

We also define position-space embeddings $I_j (j = 1, 2, 3)$ using Dirac delta basis and demonstrate that they preserve position eigenstates. This duality between frequency-space and position-space embeddings reveals a deep symmetry between quantum representations.

As an example, a geometric interpretation is introduced via the use of Frenet frames along spatial curves, allowing for the representation of localized electromagnetic waves carrying geometric signatures of trajectories. Fields such as

$$\delta_0 \circ \gamma \cdot f \circ \gamma : t \mapsto \delta_{\gamma(t)} \cdot f(\gamma(t))$$

are shown to encode the curve γ via the support and polarization f.

The relativistic Schrödinger equation for photons, in tempered distribution space, is recovered in the form

$$\hat{E}\psi = c|\hat{p}|\psi.$$

We show that in our subspace S the curl Maxwell's equations can be synthesized into the same Schrödinger's form equation

$$\hat{E}F = c|\hat{p}|F,$$

where $\hat{E}$ is the energy operator in our space W, perfectly analogous to the energy operator in the space of complex tempered distribution, $\hat{p}$ is the momentum operator in W whose magnitude operator equals the operator $\hbar$ curl on the subspace S. This shows that any wave distribution ψ, with a momentum representation vanishing around the singular plane Π, can be smoothly interpreted as encoding an electromagnetic-type field. A wave distribution ψ solves the massless relativistic Schrödinger equation if an only if the corresponding electromagnetic-type field solves the massless Schrödinger-Maxwell equation in W. Analogously, we construct a faithfull representation of the relativistic Schrödinger equation for massive particles in our space W, showing that each wave distribution state of a massive particle (complex field) can be smoothly interpreted as an electromagnetic-like field in W.

Delta distributions

$$\psi(x, t) = \delta(x \mp ct)$$

are proven to be solutions of photons equation with spectral support positive or negative, corresponding to right-moving and left-moving massless particles, respectively. The relation

$$m = \hbar|\vec{k}|/c$$

defines the relativistic mass of a photon as a function of spectral content. On the other hand, the dispersion relation of the massive plane wave fields satisfying the Maxwell-Schrödinger equation, is given by the Einstein's energy relation.

This work lays a foundation for a full spectral theory of relativistic fields within tempered distribution spaces, connecting canonical Quantum Mechanics, Maxwell's equations, and geometric field structures under a unified, mathematically rigorous umbrella.

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Infinitesimal conformal transformations and vielbein formalism

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When exploring infinitesimal transformations of differentiable manifolds, we typically use holonomic coordinate systems. However, to study spinor fields, we introduce a set of four independent vector fields $t^i_a(x)$, with $a = 0, 1, 2, 3$, defined at each point of a spacetime $(V^{1,3}, g)$. These vectors are orthonormal with respect to the spacetime metric and satisfy the condition:

$$t^i_a(x)\, t^j_b(x)\, g_{ij}(x) = \eta_{ab}, \quad \text{where} \quad \eta_{ab} = \mathrm{diag}(1, -1, -1, -1).$$

The inverse matrix $t^a_i(x)$ is defined such that:

$$t^i_a(x)\, t^a_j(x) = \delta^i_j, \quad t^i_a(x)\, t^b_i(x) = \delta^a_b.$$

This approach is known as the *vielbein formalism*, where the field $t^a_i(x)$ is called the *vielbein* [1].

The spin connection is defined by the following expression:

$$\omega_k{}^a{}_b = \left(t^i_b\, \Gamma^h_{ki} + \partial_k t^h_b\right) t^a_h. \tag{1}$$

From equation (1), we obtain the identity:

$$\partial_k t^h_a + \Gamma^h_{jk} t^j_a - \omega_k{}^b{}_a t^h_b = 0.$$

The covariant derivative of a spinor field $\psi(x)$ is given by:

$$\nabla_k \psi = \partial_k \psi - \frac{1}{4} \omega_{kab}\, \gamma^{ab} \psi = \partial_k \psi + \Gamma_k \psi,$$

where $\gamma^{ab} = \frac{1}{2}(\gamma^a \gamma^b - \gamma^b \gamma^a)$ is the antisymmetrized product of two gamma matrices.

The covariant derivative of the adjoint spinor $\overline{\psi} = \psi^\dagger \gamma^0$ is:

$$\nabla_k \overline{\psi} = \partial_k \overline{\psi} + \overline{\psi} \frac{1}{4} \omega_{kab} \gamma^{ab} = \partial_k \overline{\psi} - \overline{\psi} \Gamma_k.$$

Infinitesimal transformations of the form

$$\overline{x}^h = x^h + \epsilon \xi^h(x^1, x^2, \dots, x^n)$$

are called *conformal transformations* if the following condition is satisfied [4, p. 157]:

$$\mathfrak{L}_\xi g_{ij} = \xi_{i,j} + \xi_{j,i} = \varphi\, g_{ij}, \tag{2}$$

where $\varphi(x)$ is a scalar function.

Taking the Lie derivative of the vielbein yields:

$$\mathfrak{L}_\xi t_i^a(x) = \frac{\varphi}{2} t_i^a(x). \tag{3}$$

For any geometric object field $\Omega_M^\Lambda(\xi)$, the following identity holds [4, p. 23]:

$$\mathfrak{L}_\xi \partial_k \Omega_M^\Lambda(\xi) = \partial_k \mathfrak{L}_\xi \Omega_M^\Lambda(\xi). \tag{4}$$

Using this result, we find the Lie derivative of the spin connection:

$$\mathfrak{L}_\xi \omega_{kab} = \frac{1}{2}\left(t_{ka}\varphi_b - t_{kb}\varphi_a\right),$$

where $\varphi_b = \partial_b \varphi = t_b^j \partial_j \varphi$.

Thus, for the spin-affine connection Γ_k, we obtain:

$$\mathfrak{L}_\xi \Gamma_k = -\frac{1}{8}(t_{ka}\varphi_b - t_{kb}\varphi_a)\gamma^{ab} = -\frac{1}{4} t_{ka}\varphi_b \gamma^{ab}. \tag{5}$$

The stress-energy tensor of a spinor field ($s = \frac{1}{2}$) in the spacetime $(V^{1,3}, g)$ is given by [3]:

$$T_{jk} = \frac{i}{2}\left(\overline{\psi}\gamma_{(j}\nabla_{k)}\psi - (\nabla_{(j}\overline{\psi})\gamma_{k)}\psi\right), \tag{6}$$

where $\gamma_j = \gamma_a t_j^a(x)$.

Taking into account equations (2), (3), (4) (5), and (6), we derive the Lie derivative of the stress-energy tensor:

$$\mathfrak{L}_\xi T_{jk} = \frac{\varphi}{2}\bigg(T_{jk} - \frac{i}{4}\left(\overline{\psi}\gamma_j t_{ka}\varphi_b\gamma^{ab}\psi + \overline{\psi}\gamma_k t_{ja}\varphi_b\gamma^{ab}\right.$$
$$\left. + \overline{\psi} t_{ka}\varphi_b\gamma^{ab}\gamma_j + \overline{\psi} t_{ja}\varphi_b\gamma^{ab}\gamma_k\psi\right)\bigg).$$

There exists a scalar quantity:

$$|A|^2 = A^i g_{ij} A^j = (\overline{\psi}\gamma^i\psi)\, g_{ij}\, (\overline{\psi}\gamma^j\psi),$$

where $A^i = \overline{\psi}\gamma^i\psi$ is the four-current of the spinor field ψ.

This scalar is invariant under conformal transformations:

$$\mathfrak{L}_\xi(|A|^2) = \mathfrak{L}_\xi\left(A^i g_{ij} A^j\right) = \mathfrak{L}_\xi\left(\overline{\psi}\gamma^i\psi \cdot g_{ij} \cdot \overline{\psi}\gamma^j\psi\right) = 0.$$

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On topologization of subsemigroups of the bicyclic monoid

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In this paper we shall follow the terminology of [2, 5, 6, 7].

A semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$.

A topology τ on a semigroup S is called a *semigroup* (*shift-continuous*) topology if (S, τ) is a topological (semitopological) semigroup.

The bicyclic monoid $\mathcal{C}(p, q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq = 1$. The bicyclic monoid admits only the discrete semigroup Hausdorff topology [4]. Bertman and West in [1] extended this result for the case of Hausdorff semitopological semigroups. T_1-topologizations of the bicyclic monoid $\mathcal{C}(p, q)$ are studied in [3].

Theorem 1. *Let S be a subsemigroup of the bicyclic semigroup $\mathscr{C}(p, q)$. If S contains infinitely many idempotents then every shift-continuous Hausdorff topology on S is discrete.*

Corollary 2. *Let S be an inverse subsemigroup of the bicyclic semigroup $\mathscr{C}(p, q)$. Then every shift-continuous Hausdorff topology on S is discrete.*

Also we give sufficient algebraic conditions on a subsemigroup S of the bicyclic semigroup $\mathscr{C}(p, q)$ when the semigroup S admits a non-discrete Hausdorff semigroup (shift-continuous) topology.

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Isoperimetric profile and quantitative orbit equivalence for lamplighter-like groups (joint work with Vincent Dumoncel)

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Measure equivalence has been introduced by Gromov as a measured analogue of quasi-isometry. In this talk we focus on the closely related notion of orbit equivalence which is in fact a source of examples for measure equivalence.

Two groups G and H are orbit equivalent if there exist two free probability measure-preserving G- and H-actions on a standard probability space, having the same orbits.

However Orstein and Weiss proved that two infinite amenable groups are orbit equivalent. To get an interesting theory, we need to strengthen the definition of orbit equivalence.

In this talk, we introduce the quantitative versions of orbit equivalence, which propose to add some restrictions on two maps called *cocycles*, which describe more precisely the orbit equalities of a given orbit equivalence between G and H. Given maps $\varphi, \psi \colon \mathbb{R}_+ \to \mathbb{R}_+$, we define the notion of φ-integrability of a cocycle (for instance, being L^p when $\varphi(x) = x^p$), and the notion of (φ, ψ)-integrability for an orbit equivalence (which asks that one cocycle is φ-integrable and the other is ψ-integrable). If G and H are amenable, these quantifications provide interesting information on their geometry, since the *isoperimetric profiles* of the groups give obstructions to the existence of quantitative versions of orbit equivalence (see [1, Theorems 1.1, Corollary 4.7]), and then lead to the following problem.

Problem 1. What is the "highest" map $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that there exists a (φ, L^0)-integrable orbit equivalence from G to H, and vice versa?

In some sense, this is a more quantitative comparison between groups. The highest quantification we can get answers to the following question: if two groups are not quasi-isometric, how much do their geometry differ?

In a joint work with Vincent Dumoncel, we study quantitative orbit equivalence for *lampshuffler groups*. Given a group H, the lampshuffler group over H is

$$\mathsf{Shuffler}(H) := \mathrm{FSym}(H) \rtimes H,$$

where $\mathrm{FSym}(H)$ is the set of permutations of H of finite support, and the action of H on it is given by $k \cdot \sigma \colon h \in H \to k\sigma(k^{-1}h)$.

Lampshufflers belong to a large class of groups which look like *lamplighter group*. They have been intensively studied in [2], where the authors found conditions for two lamplighters to be quasi-isometric, for two lampshufflers to be quasi-isometric, etc.

Outlines of our work: Our goal is to quantitatively compare lampshufflers. We first build explicit orbit equivalence couplings between lampshufflers and quantify the associated cocycles. Secondly, we compute the isoperimetric profiles of lampshufflers to prove that the quantifications we find are optimal. In this talk, I will present our results with more details.

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Local Moduli of Sasaki-Einstein metrics on rational homology 7-spheres

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For Sasakian manifolds, which are roughly speaking the odd dimensional analogue of Kähler manifolds, the moduli problem has been addressed and an appropriate notion of moduli space has been achieved, see [1] and references therein. In particular, finding the number of components of this moduli has been studied and used to obtain lower bounds for the dimension of the moduli space for links of Brieskorn-Pham polynomials and Smale manifolds [4, 6, 2, 7]. An important ingredient to obtain information on the moduli is given by the use of invertible polynomials, particularly to describe the local moduli of Sasaki-Einstein metrics.

In this talk, which is based on a joint work with J. Lope [10], we determine the dimension of the local moduli space of Sasaki-Einstein metrics for links of invertible polynomials coming from the list of Johnson and Kollár of anticanonically embedded Fano 3-folds of index 1 [5] that produce $\mathbb{Q}$-homology 7-spheres, that is, 7-manifolds whose $\mathbb{Q}$-homology equals that of the 7-sphere [3, 8]. In order to do so, we propose additional conditions to the Diophantine equations associated to this problem. We also find solutions for the problem associated to the moduli for the Berglund-Hübsch duals [9] of links arising from Thom-Sebastiani sums of chain and cycle polynomials.

Our findings can be interpreted in two different settings:

- Seifert S^1-bundles are $\mathbb{Q}$-homology spheres if and only if the corresponding orbifolds are $\mathbb{Q}$-homology complex projective spaces, so our results describes some components of the moduli space of $\mathbb{Q}$-homology complex projective 3-spaces with quotient singularities.
- Sasaki-Einstein structures on the manifold determine Ricci-flat Kähler cone metrics on the corresponding affine cone, so our results give information on the moduli of Calabi-Yau cones.

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On boundary behavior of unclosed mappings with moduli inequality

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Let D be a domain in $\overline{\mathbb{R}^n}$ and let $b \in \partial D$. Then D has property P_1 at b if the following condition is satisfied: If E and F are connected subsets of D such that $b \in \overline{E} \cup \overline{F}$, then $M(\Gamma(E, F, D)) = \infty$, where M denotes the (conformal) modulus of families of paths in $\mathbb{R}^n$ (see the definition below), and $\Gamma(E, F, D)$ is a family of paths joining E and F in D (see e.g. [1, Definition 17.5]). The following results hold.

Theorem A. *Suppose that $f : D \to D'$ is a quasiconformal mapping and that D has property P_1 at $\in \partial D$. Then $C(f, b)$ contains at most one point at which D' is finitely connected (see* [1, Theorem 17.13]*).*

Theorem B. *Let $f : D \to \mathbb{R}^n$ be quasiregular mapping with $C(f, \partial D) \subset \partial f(D)$. If D is locally connected at a point $b \in \partial D$ and $D' = f(D)$ is qc accessible at some point $y \in C(f, b)$, then $C(f, b) = \{y\}$ (see e.g.* [2, Theorem 4.2]*, cf.* [3, Theorem 4.2]*).*

We give some generalization of Theorems **A** and **B**. Recall some definitions. A Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for the family Γ of paths γ in $\mathbb{R}^n$, if the relation $\int\limits_{\gamma} \rho(x)\,|dx| \geqslant 1$ holds for all (locally rectifiable) paths $\gamma \in \Gamma$. In this case, we write: $\rho \in \operatorname{adm} \Gamma$. Let $p \geqslant 1$, then *p-modulus* of Γ is defined by the equality $M_p(\Gamma) = \inf\limits_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^p(x)\,dm(x)$. Let $x_0 \in \mathbb{R}^n$, $0 < r_1 < r_2 < \infty$,

$$S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}\,, \quad B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\} \tag{1}$$

and $A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$. Let $S_i = S(x_0, r_i)$, $i = 1, 2$, where spheres $S(x_0, r_i)$ centered at x_0 of the radius r_i are defined in (1). Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function satisfying the condition $Q(x) \equiv 0$ for $x \in \mathbb{R}^n \setminus D$. Let $p \geqslant 1$. Due to [4], a mapping $f : D \to \overline{\mathbb{R}^n}$ is called a *ring Q-mapping at the point $x_0 \in \overline{D} \setminus \{\infty\}$ with respect to p-modulus*, if the condition

$$M_p(f(\Gamma(S_1, S_2, D))) \leqslant \int\limits_{A \cap D} Q(x) \cdot \eta^p(|x - x_0|)\,dm(x) \tag{2}$$

holds for some $r_0(x_0)>0$, all $0<r_1<r_2<r_0$ and all Lebesgue measurable functions $\eta:(r_1,r_2)\rightarrow [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2}\eta(r)\,dr\geqslant 1\,. \tag{3}$$

Recall that a mapping $f:D\rightarrow{\Bbb R}^n$ is called *discrete* if the pre-image $\{f^{-1}\left(y\right)\}$ of each point $y\,\in\,{\Bbb R}^n$ consists of isolated points, and *is open* if the image of any open set $U\subset D$ is an open set in ${\Bbb R}^n$. Later, in the extended space $\overline{{\Bbb R}^n}={\Bbb R}^n\cup\{\infty\}$ we use the *spherical (chordal) metric* $h(x,y)=|\pi(x)-\pi(y)|$, where π is a stereographic projection $\overline{{\Bbb R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1},\frac{1}{2})$ in ${\Bbb R}^{n+1}$, namely,

$$h(x,\infty)=\frac{1}{\sqrt{1+{|x|}^2}}\,,\quad h(x,y)=\frac{|x-y|}{\sqrt{1+{|x|}^2} \sqrt{1+{|y|}^2}}\,, \quad x\ne \infty\ne y$$

(see [1, Definition 12.1]). Further, the closure $\overline{A}$ and the boundary ∂A of the set $A\subset\overline{{\Bbb R}^n}$ we understand relative to the chordal metric h in $\overline{{\Bbb R}^n}$. Given a mapping $f:D\rightarrow {\Bbb R}^n$, we denote $C(f, x):=\{y\in \overline{{\Bbb R}^n}:\exists\,x_k\in D: x_k\rightarrow x, f(x_k)\rightarrow y, k\rightarrow\infty\}$ and $C(f, \partial D)=\bigcup\limits_{x\in \partial D}C(f, x)\,.$ In what follows, $\mathrm{Int\,}A$ denotes the set of inner points of the set $A\subset \overline{{\Bbb R}^n}$. Recall that the set $U\subset\overline{{\Bbb R}^n}$ is neighborhood of the point z_0, if $z_0\in \mathrm{Int\,}A$. Due to [4], we say that a function $\varphi:D\rightarrow{\Bbb R}$ has a *finite mean oscillation* at a point $x_0\in D$, write $\varphi\in FMO(x_0)$, if $\limsup\limits_{\varepsilon\rightarrow 0}\frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(x_0,\,\varepsilon)} |{\varphi}(x)-\overline{{\varphi}}_{\varepsilon}|\ dm(x)<\infty$, where $\overline{{\varphi}}_{\varepsilon}=\frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(x_0,\,\varepsilon)} {\varphi}(x) dm(x)\,.$ Let $Q:{\Bbb R}^n\rightarrow [0,\infty]$ be a Lebesgue measurable function. We set $Q^{\,\prime}(x)=\left\{\begin{array}{rr} Q(x), & Q(x)\geqslant 1\,,\\ 1, & Q(x)<1\,.\end{array} \right.$ Denote by $q^{\,\prime}_{x_0}$ the mean value of $Q^{\,\prime}(x)$ over the sphere $|x-x_0|=r$, that means,

$$q^{\,\prime}_{x_0}(r):=\frac{1}{\omega_{n-1}r^{n-1}}\int\limits_{|x-x_0|=r}Q^{\,\prime}(x)\,d\mathcal{H}^{n-1}\,. \tag{4}$$

Note that, using the inversion $\psi(x)=\frac{x}{|x|^2}$, we may give the definition of FMO as well as the quantity in (4) for $x_0=\infty$. We say that the boundary ∂D of a domain D in ${\Bbb R}^n$, $n\geqslant 2$, is *strongly accessible at a point $x_0\in \partial D$ with respect to the p-modulus* if for each neighborhood U of x_0 there exist a compact set $E\subset D$, a neighborhood $V\subset U$ of x_0 and $\delta>0$ such that $M_p(\Gamma(E,F, D))\geqslant \delta$ for each continuum F in D that intersects ∂U and ∂V.

Theorem 1. *([5]). Let $p\geqslant 1$, let D and $D^{\,\prime}$ be domains in ${\Bbb R}^n$, $n\geqslant 2$, $f:D\rightarrow D^{\,\prime}$ be an open discrete mapping satisfying relations (2)–(3) at the point $b\in \partial D$ with respect to p-modulus, $f(D)=D^{\,\prime}$. In addition, assume that 1) the set $E:=f^{\,-1}(C(f, \partial D))$ is nowhere dense in D and D is finitely connected on E, i.e., for any $z_0\in E$ and any neighborhood $\widetilde{U}$ of z_0 there is a neighborhood $\widetilde{V}\subset \widetilde{U}$ of z_0 such that $(D\cap \widetilde{V})\setminus E$ consists of finite number of components; 2) for any neighborhood U of b there is a neighborhood $V\subset U$ of b such that: 2a) $V\cap D$ is connected, 2b) $(V\cap D)\setminus E$ consists at most of m components, $1\leqslant m<\infty$, 3) $D^{\,\prime}\setminus C(f, \partial D)$ consists of finite components, each of them has a strongly accessible boundary with respect to p-modulus. Suppose that at least one of the following conditions is satisfied: $4_1)$ a function Q has a finite mean oscillation at the* *point* b; 4_2) $q_b(r) = O\left(\left[\log\frac{1}{r}\right]^{n-1}\right)$ *as* $r \to 0$; 4_3) *the condition* $\int\limits_0^{\delta(b)} \frac{dt}{t^{\frac{n-1}{p-1}} {q_b^{\prime}}^{\frac{1}{p-1}}(t)} = \infty$ *holds for some* $\delta(b) > 0$. *Then* f *has a continuous extension to* b.

If the above is true for any point $b \in \partial D$, *the mapping* f *has a continuous extension* $\overline{f} : \overline{D} \to \overline{D'}$, *moreover,* $\overline{f}(\overline{D}) = \overline{D'}$.

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Bonded Knots and Braids

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Proteins, fundamental to biological function, are complex molecules composed of amino acid chains that fold into highly specific three-dimensional configurations. These folded structures are stabilized by intramolecular bonds—interactions between distant residues, that are essential for maintaining shape and functionality. Mathematically, such structures can be modeled as bonded knots, where the protein backbone forms a knot or open curve, and the stabilizing interactions are represented by bonds connecting non-adjacent segments ([2, 3]).

In this talk, I will present the theory of bonded knots and its extension to bonded braids, emphasizing their structural, topological, and algebraic features. Bonded knots ([1]) generalize classical knot theory by introducing bond constraints, which fall into three main classes: long bonds (topological and rigid-vertex), regular bonds (with unknotted connections), and tight bonds (modeled as non-crossing line segments). For each type, I will describe a system of Reidemeister-type moves—both in the topological and rigid frameworks—and introduce core invariants that classify these objects.

I will then transition to bonded braids ([4]), discussing an Alexander-type theorem that relates bonded knots to their braid representations in the topological category. The talk will include the definition of the bonded braid monoid, its generating set and relations, and a Markov-type theorem capturing braid equivalence. I will also sketch how this monoid embeds into a group, revealing deeper algebraic structure.

Time permitting, I will conclude with a look at bonded knots on the torus and their relation to doubly periodic bonded tangles ([5, 6, 7]), offering insights into their covering spaces and potential relevance to structural biology. This presentation provides an accessible entry point into the emerging theory of bonded knots and braids and its rich connections to topology, algebra, and the geometry of biomolecular systems.

This is a joint work with Prof. Dr. L.H. Kauffman (University of Illinois at Chicago, U.S.A.), Prof. Dr. Sofia Lambropoulou (National Technical University of Athens, Greece) and Dr. Sonia Mahmoudi (Tohoku University, Japan).

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Minkowski chirality

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In his ground-laying work [1], Grünbaum set up a general framework for quantifying the point (a)symmetry of convex bodies, i.e., compact convex sets with nonempty interior. Specifically, a measure of (a)symmetry is a similarity-invariant (or even affinely invariant) Hausdorff continuous function f that takes convex bodies to the unit interval with the property that $f(K) = 1$ if and only if K is point-symmetric.

In [1], some generalizations are discussed, for example quantifying (a)symmetry with respect to reflections across affine subspaces of dimension at least one. However, the author mentions lack of results in the literature in this direction. Different notions of chirality or axiality for quantifying the (a)symmetry of planar shapes with respect to reflections across straight lines have been investigated in the mathematical literature in the past decades. Asymmetry notions for planar convex bodies are also studied in mathematical chemistry , where polygons serve as abstractions of molecules and where chirality impacts chemical properties.

Our contribution is based on an extension of the notion of Minkowski asymmetry, which, for a convex body K, is defined as the smallest dilation factor $\lambda > 0$ such that K is a subset of a translated and dilated copy of $-K$, the mirror image of K upon reflection across the coordinate origin. We incorporate reflections across higher-dimensional (affine) subspaces by defining the *jth Minkowski chirality* $\alpha_j(K)$ as the smallest dilation factor $\lambda > 0$ such that the convex body $K \subset \mathbb{R}^n$ is a subset of a translated and dilated copy of $\Phi_U(K)$, where Φ_U denotes the *reflection* across the j-dimensional affine subspace $U \subset \mathbb{R}^n$ for $j \in \{0, \ldots, n\}$. Note that the Minkowski asymmetry is $\alpha_0(K)$ in this terminology.

It is well-known that $\alpha_0(K) \in [1, n]$ for all convex bodies $K \subset \mathbb{R}^n$, with $\alpha_0(K) = 1$ if and only if K is point-symmetric, and $\alpha_0(K) = n$ if and only if K is a fulldimensional simplex, see [1].

Our main result for convex bodies in general dimensions extends the upper bound on the Minkowski asymmetry to all Minkowski chiralities $\alpha_j(K)$ for any $j \in \{0, \ldots, n\}$.

Theorem 1. *Let $K \subset \mathbb{R}^n$ be a convex body and $j \in \{0, \ldots, n\}$. Then*

$$1 \leq \alpha_j(K) \leq \min\{n, \frac{\alpha_0(K) + 1}{2}\sqrt{n}\},$$

with $\alpha_j(K) = 1$ if and only if there exists a j-dimensional affine subspace U such that $K = \Phi_U(K)$.

In fact, the upper bound in 1 can be strengthened and unified to

$$\alpha_j(K) \leq \sqrt{\alpha_0(K)n} \tag{1}$$

for any convex body $K \subset \mathbb{R}^n$ and $j \in \{0, \ldots, n\}$. Since $\alpha_0(K) \leq n$ with $\alpha_0(K) = n$ solely for simplices, this result implies $\alpha_j(K) \leq n$ and in particular that only simplices might have jth Minkowski chirality n.

We recall that the *Banach–Mazur distance* between convex bodies $K, L \subset \mathbb{R}^n$ is defined by

$$d_{BM}(K, L) = \inf\{\lambda > 0 : t^1 + K \subset A(L) \subset t^2 + \lambda K,\ A \in GL(\mathbb{R}^n),\ t^1, t^2 \in \mathbb{R}^n\},$$

where $GL(\mathbb{R}^n)$ denotes the set of invertible real $n \times n$ matrices.

The inequality (1) is also consequential for the absolute upper bound on the jth Minkowski chirality. Any convex body K with Minkowski asymmetry $\alpha_0(K)$ near n is close to a simplex in the Banach-Mazur distance. Together with (1), this means that either the supremum of $\alpha_j(T)$ over all simplices $T \subset \mathbb{R}^n$ equals n, or there exists some constant $c(n, j) < n$ such that any convex body $K \subset \mathbb{R}^n$ satisfies $\alpha_j(K) \leq c(n, j)$. In other words, we can determine whether the inequality $\alpha_j(K) \leq n$ is tight by checking only simplices.

Although this remains a challenging problem in general, we are able to solve it in the planar case for the first Minkowski chirality.

Theorem 2. *Let $K \subset \mathbb{R}^2$ be a triangle. Then the infimum in the definition of $\alpha_1(K)$ is attained at some affine subspace U of $\mathbb{R}^2$ that is necessarily*

(1) parallel to the bisector of one of the largest interior angles of K,
(2) parallel to the bisector of one of the smallest interior angles of K, or
(3) perpendicular to one of the longest edges of K.

Moreover, we have when $K \subset \mathbb{R}^2$ is a triangle

$$\alpha_1(K) = \left[1, \sqrt{2}\right), \tag{2}$$

with $\alpha_1(K) = 1$ precisely for isosceles triangles.

The question of how large $\alpha_j(K)$ can be for general n and j is still open, as even deciding whether the inequality $\alpha_j(K) \leq n$ is actually tight appears to be difficult. Instead, we focus on a special class of convex bodies and answer the first question for planar point-symmetric convex bodies: the upper bound from 1 becomes $\sqrt{2}$ in this case, and the following two theorems show that this bound is reached precisely by a specific family of parallelograms.

The second theorem uses the *John ellipsoid* $\mathcal{E}_J(K)$ of a convex body $K \subset \mathbb{R}^n$, which is the unique volume-maximal ellipsoid contained in K.

Theorem 3. *Let $K \subset \mathbb{R}^2$ be a point-symmetric convex body with $d_{BM}(K, P) \geq 1 + \epsilon$ for a parallelogram $P \subset \mathbb{R}^2$ and some $\epsilon > 0$. Then*

$$\alpha_1(K) < \sqrt{2}\left(1 - \frac{\epsilon}{10}\right).$$

Theorem 4. *Let $K \subset \mathbb{R}^2$ be a parallelogram. Then the infimum in the definition of $\alpha_1(K)$ is attained at some affine subspace U of $\mathbb{R}^2$ that is necessarily parallel to*

(1) the bisector of an angle formed by consecutive edges of K,
(2) the bisector of an angle formed by the diagonals of K, or
(3) a principal axis of the John ellipse $\mathcal{E}_J(K)(K)$ of K.

Moreover, we have when $K \subset \mathbb{R}^2$ is a parallelogram

$$\alpha_1(K) = \left[1, \sqrt{2}\right], \tag{3}$$

with $\alpha_1(K) = 1$ precisely for rectangles and rhombuses. Moreover, $\alpha_1(K) = \sqrt{2}$ if and only if the angles between the diagonals coincide with the interior angles and the ratio between the lengths of the longer edges and the shorter edges is at least $\sqrt{2}$.

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Forbidden Four Cycle in Diametrical Graphs and Embedding in Stars

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The ultrametric spaces generated by arbitrary nonnegative vertex labelings on both finite and infinite trees were first considered in [2] and studied in [5, 4]. The simplest types of infinite trees are rays and star graphs. The totally bounded ultrametric spaces generated by labeled almost rays have been characterized in [7]. Furthermore, paper [6] contains a purely metric characterization of ultrametric spaces generated by labeled star graphs.

Our main purpose is to give an answer to the following problem.

Problem 1. Let (X, d) be an ultrametric space. Find conditions under which (X, d) admits an isometric embedding in an ultrametric space generated by labeled star graph.

Here and in what follows, by *labeled* star graph $S(l)$ we will mean a star graph S equipped with a labeling $l\colon V(S) \to \mathbb{R}^+$, where $V(S)$ is the vertex set of the star graph S.

Let $S(l)$ be a labeled star graph. As in [2] we define a mapping $d_l\colon V(S) \times V(S) \to \mathbb{R}^+$ by

$$d_l(u,v) = \begin{cases} 0, & \text{if } u = v, \\ \max\limits_{w \in V(P)} l(w), & \text{otherwise,} \end{cases}$$

where P is the path joining u and v in S. Let (Y, ρ) be an ultrametric space. We say that (Y, ρ) is generated by labeled star graph $S(l)$ if Y is the vertex set of S and the equality $\rho = d_l$ holds.

We will also use the concept of diametrical graph introduced in [1]. The next definition is a modification of Definition 2.1 from [8].

Definition 2. Let (X, d) be an ultrametric space with $\operatorname{card} X \geq 2$. A graph G is called the diametrical graph of (X, d) if X is the vertex set of G and points $x, y \in X$ are adjacent in G if and only if

$$d(x,y) = \sup\{d(u,v)\colon u,\, v \in X\}.$$

The following theorem gives a solution of Problem 1.

Theorem 3. *Let (X, d) be an infinite ultrametric space. Then the following statements are equivalent:*

(i) *There is (Y, ρ) such that (Y, ρ) is generated by labeled star graph and (X, d) is isometric to a subspace of (Y, ρ).*

(ii) *(X, d) contains no four-point subspace with diametrical graph isomorphic to the cycle C_4.*

If (X, d) is a compact ultrametric space, then Theorem 3 follows from Theorem 5.2 of [3].

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Spray-Invariant Sets in Infinite-Dimensional Manifolds

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This talks investigates the behavior of geodesics within subsets of infinite-dimensional manifolds, including singular spaces such as stratified spaces. We define *spray-invariant* sets as those where any geodesic starting within the set remains entirely within it. The regularity of these sets significantly impacts their geometric properties.

We study these sets in the context of spray geometry, without relying on Finsler or Riemannian metrics, enabling the analysis of geodesic dynamics in infinite-dimensional manifolds where traditional geometric tools may not be applicable. For a subset S of a manifold M and a spray $\mathbf{S}$ on M, we define an *admissible set* $A_{\mathbf{S},S}$ that characterizes when a geodesic remains within S. We prove that if S is closed, then a geodesic lies entirely in S if and only if its tangent vector belongs to $A_{\mathbf{S},S}$ for all time, establishing $A_{\mathbf{S},S}$ as a key invariant. For sufficiently differentiable submanifolds S, we show that $A_{\mathbf{S},S}$ characterizes totally geodesic submanifolds.

We also show that spray-invariant sets remain invariant under spray automorphisms. We explore the relationship between spray invariance and the tangency of the spray to the admissible set, addressing this using the Nagumo-Brezis Theorem, where we establish the equivalence between spray invariance and this tangency condition.

Finally, we study Lie group actions on Banach manifolds and their orbit type decompositions. We prove that if the action admits suitable local slices (defined by invariance, local triviality, and transversality), then each orbit type stratum is invariant under a group-invariant spray.

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Constant mean curvature surfaces with harmonic Gauss maps in three-dimensional Lie groups

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Let G be a $(n+1)$-dimensional Lie group with a left invariant metric. For an (immersed) hypersurface M in G define the Gauss map Φ of M by

$$\Phi\colon M \to \mathbb{R}P^n; \Phi(p) = dL_{p^{-1}}(N_pM).$$

Here a point p is identified with its image under the immersion, N_pM is a normal space of N at p, and $dL_{p^{-1}}$ is the differential of the left translation in G. If M is orientable we can consider also the orientable Gauss map whose target space is S^n. It was shown in [1] that if the metric of G is biinvariant then Φ is harmonic if and only if M is of constant mean curvature (CMC). This is a generalization of a classical Euclidean result of [4]. In particular, for $n = 2$ biinvariant metrics exist on the simply-connected Lie groups $\mathbb{R}^3$ and $\mathbb{S}^3$ and are their usual metrics of constant curvature. It appears that for many other classes of left invariant metrics the equivalence between the harmonicity of Φ and CMC does not take place. For example, each CMC hypersurface with the harmonic Gauss map in the $(2m+1)$-dimensional Heisenberg group is locally a vertical cylinder ([3]).

Using a well-known description from [2] of left invariant metrics on a three-dimensional Lie group G, we derive criteria of the harmonicity of Φ for a general such metric. This allows us to prove the following:

Theorem 1. *Let a left invariant metric on a connected three-dimensional unimodular Lie group G be right invariant with respect to a one-dimensional subgroup $H \subset G$, but not biinvariant, and let M be a connected surface in G. Then from any two of the following claims the third follows:*

(1) M is CMC;
(2) the Gauss map of M is harmonic;
(3) M is either everywhere orthogonal to the one-dimensional foliation generated by H (is horizontal) or is composed of leaves of this foliation (is vertical).

This applies to all left invariant metrics on the 3-dimensional Heisenberg group, some metrics on the groups E(2) (of orientation-preserving Euclidean plane isometries), $\mathrm{SL}(2,\mathbb{R})$ and their universal coverings, and to some non-biinvariant metrics on SU(2) and its universal covering $\mathbb{S}^3$. It allows us to give explicit descriptions of CMC surfaces with harmonic Gauss maps for some model metrics on these groups. We also consider some examples of metrics that neither are biinvariant nor satisfy the conditions of the theorem 1 (in particular, any left invariant metric on the Lie group Sol is like that).

We also use the Gauss map harmonicity criteria for non-unimodular groups to prove the following:

Theorem 2. *A complete connected surface in the hyperbolic space $\mathbb{H}^3$ is CMC with the harmonic Gauss map (in the Lie group sense) if and only if it is a horosphere parallel to the sphere at infinity.*

Here the Lie group structure on the half-space $(z > 0)$ model of $\mathbb{H}^3$ with the usual metric $\frac{1}{z^2}(dx^2+dy^2+dz^2)$ corresponds to the orthonormal frame of left-invariant fields $X_1 = z\frac{\partial}{\partial x}$, $X_2 = z\frac{\partial}{\partial y}$, $X_3 = z\frac{\partial}{\partial z}$, and the horospheres are thus of the form $z = z_0$.

These results were partially obtained in a joint work with Iryna Savchuk.

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On controllability problems for the heat equation on a half-plane controlled by the Neumann boundary condition with a point-wise control

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Consider the following control system on a half-plane

$$w_t = \Delta w, \qquad x_1 \in \mathbb{R}_+,\ x_2 \in \mathbb{R},\ t \in (0,T), \tag{1}$$

$$w_{x_1}\big(0,(\cdot)_{[2]},t\big) = \delta_{[2]}u(t), \qquad x_2 \in \mathbb{R},\ t \in (0,T), \tag{2}$$

$$w\big((\cdot)_{[1]},(\cdot)_{[2]},0\big) = w^0, \qquad x_1 \in \mathbb{R}_+,\ x_2 \in \mathbb{R}, \tag{3}$$

where $\mathbb{R}_+ = (0,\infty)$, $T > 0$, $u \in L^\infty(0,T)$ is a control, $\delta_{[m]}$ is the Dirac distribution with respect to x_m, $m = 1,2$, $w^0 \in L^2(\mathbb{R}_+ \times \mathbb{R})$. The subscripts [1] and [2] associate with the variable numbers, e.g., $(\cdot)_{[1]}$ and $(\cdot)_{[2]}$ correspond to x_1 and x_2, respectively, if we consider $f(x)$, $x \in \mathbb{R}^2$. This control system is considered in spaces of Sobolev type. We treat equality (2) as the value of the distribution w_{x_1} on the line $x_1 = 0$.

A point-wise control is a mathematical model of a source supported in a domain of very small size with respect to the whole domain. That is why studying control problems under a point-wise control is an important issue in control theory.

The controllability problems for the heat equation on a half-plane controlled by the Neumann boundary condition with a point-wise control is studied. These problems for the heat equation on a half-plane controlled by the Dirichlet boundary condition with a point-wise control were studied in [1].

Let $w^0 \in L^2(\mathbb{R}_+ \times \mathbb{R})$. By $\mathcal{R}_T(w^0)$, denote the set of all states $w^T \in L^2(\mathbb{R}_+ \times \mathbb{R})$ for which there exists a control $u \in L^\infty(0,T)$ such that there exists a unique solution w to system (1)–(3) such that $w\big((\cdot)_{[1]},(\cdot)_{[2]},T\big) = w^T$.

Definition 1. A state $w^0 \in L^2(\mathbb{R}_+ \times \mathbb{R})$ is said to be controllable to a target state $w^T \in L^2(\mathbb{R}_+ \times \mathbb{R})$ in a given time $T > 0$ if $w^T \in \mathcal{R}_T(w^0)$.

Definition 2. A state $w^0 \in L^2(\mathbb{R}_+ \times \mathbb{R})$ is said to be approximately controllable to a target state $w^T \in L^2(\mathbb{R}_+ \times \mathbb{R})$ in a given time $T > 0$ if $w^T \in \overline{\mathcal{R}_T(W^0)}$, where the closure is considered in the space $L^2(\mathbb{R}_+ \times \mathbb{R})$.

For control system (1)–(3), the set $\mathcal{R}_T(0) \subset L^2(\mathbb{R}_+ \times \mathbb{R})$ of its states reachable from 0 (i.e. the set which is formed by the end states $w(\cdot,T)$ of this system when controls $u \in L^\infty(0,T)$) and the set $\mathcal{R}_T^L(0) \subset \mathcal{R}_T(0) \subset L^2(\mathbb{R}_+ \times \mathbb{R})$ of its states reachable from 0 by using the controls $u \in L^\infty(0,T)$ satisfying the restriction $\|u\|_{L^\infty(0,T)} \le L$ (where $L > 0$ is a given constant) are studied to investigate the (approximate) controllavility properties. It is established that a function $f \in \mathcal{R}_T(0)$ can be represented in the form $f(x) = g\big(|x|^2\big)$ a.e. in $\mathbb{R}_+ \times \mathbb{R}$ where $g \in L^2(0,+\infty)$. In fact, the problem dealing with functions from $L^2(\mathbb{R}_+ \times \mathbb{R})$ is reduced to a problem dealing with functions from $L^2(0, +\infty)$. To this aid, operators Ψ and Φ are introduced and studied (see below).

If $f \in L^2(\mathbb{R}_+ \times \mathbb{R})$ and $f(x) = g(|x|^2)$, $x \in \mathbb{R}_+ \times \mathbb{R}$, for some g defined on $\mathbb{R}_+$, then $g \in L^2(\mathbb{R}_+)$ and $\|f\|_{L^2(\mathbb{R}_+\times\mathbb{R})} = \sqrt{\frac{\pi}{2}}\|g\|_{L^2(\mathbb{R}_+)}$ holds; and vice versa: if $g \in L^2(\mathbb{R}_+)$, then for $f(x) = g(|x|^2)$, $x \in \mathbb{R}_+ \times \mathbb{R}$, we have $f \in L^2(\mathbb{R}_+ \times \mathbb{R})$. Taking this into account, we can introduce the space

$$\mathcal{H} = \left\{ f \in L^2(\mathbb{R}_+ \times \mathbb{R}) \;\middle|\; \exists g \in L^2(\mathbb{R}_+) \quad f(x) = g(|x|^2) \text{ a.e. on } \mathbb{R}_+ \times \mathbb{R} \right\} \tag{4}$$

and the operator $\Psi : \mathcal{H} \to L^2(\mathbb{R}_+)$ with the domain $D\big(\Psi\big) = \mathcal{H}$ for which

$$\Psi f = g \Leftrightarrow \big(f(x) = g(|x|^2) \text{ a.e. on } \mathbb{R}_+ \times \mathbb{R}, \quad f \in D\big(\Psi\big) = \mathcal{H}.$$

One can see that Ψ is invertible, $\Psi^{-1} : L^2(\mathbb{R}_+) \to \mathcal{H}$, and $\big(\Psi^{-1}g\big)(x) = g(|x|^2)$, $x \in \mathbb{R}_+ \times \mathbb{R}$ for $g \in D\big(\Psi^{-1}\big) = L^2(\mathbb{R}_+)$.

Thus, Ψ is an isomorphism of $\mathcal{H}$ and $L^2(\mathbb{R}_+)$, and $\|\Psi\| = \sqrt{\frac{2}{\pi}}$. Moreover, $\mathcal{H}$ is a Hilbert space with respect to the inner product $\langle\cdot,\cdot\rangle_{L^2(R_+\times\mathbb{R})}$ and $2\langle f, h\rangle_{L^2(R_+\times\mathbb{R})} = \pi\langle\Psi f, \Psi h\rangle_{L^2(\mathbb{R}_+)}$, $f \in \mathcal{H}$, $h \in \mathcal{H}$.

Let us introduce the operator $\Phi : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ with the domain $D(\Phi) = L^2(\mathbb{R}_+)$ by the rule

$$\big(\Phi g\big)(\rho) = \frac{1}{2} \lim_{N\to\infty} \int_0^N g(r) J_0\big(\sqrt{r\rho}\big)\, dr, \quad \rho \in \mathbb{R}_+,\ g \in L^2(\mathbb{R}_+),$$

where J_0 is the Bessel function of order 0. We prove that Φ is invertible and $\Phi^{-1} = \Phi$, in particular, Φ is an isometric isomorphism of $L^2(\mathbb{R}_+)$. Note that the transform providing by the operator Φ is a modification of the well-known Hankel transform of order 0.

The operators Ψ and Φ are key tools of this work, which allow to obtain the following main results:

- **(a)** some properties of the set $\mathcal{R}_T(0)$, in particular, $\overline{\mathcal{R}_T(0)} = \mathcal{H}$;
- **(b)** some properties of the set $\mathcal{R}_T^L(0)$;
- **(c)** necessary and sufficient conditions for controllability in a given time under the control bounded by a given constant;
- **(d)** sufficient conditions for approximate controllability in a given time under the control bounded by a given constant;
- **(e)** necessary and sufficient conditions for approximate controllability in a given time, in particular, the origin can be driven to a given state $w^T \in L^2(R_+ \times \mathbb{R})$ in a given time T iff $w^T \in \mathcal{H}$;
- **(f)** the lack of controllability to the origin.

Results **(c)** and **(d)** are obtained from **(b)**, and result **(e)** follows from **(a)**. The method of obtaining result **(f)** is very similar to that in [3]. The results are illustrated by examples.

The main results of the present paper are rather similar to those of [1]. However, the methods of obtaining them are essentially different in these two papers. Roughly speaking, we deal with the two-dimensional case studying reachability sets and constructing the solutions to controllability and approximate controllability problems in [1], but reducing the two-dimensional reachability sets to the one-dimensional ones, we deal with the one-dimensional case studying these problems and constructing their solutions in the present paper. In addition, the methods used to study the one-dimensional reachability sets in this paper principally differ from those used for two-dimensional sets in [1]. Moreover, some results of the present work have not analogues in [1]. Most of the obtained results were published in [2].

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Normal forms of Morse-Bott functions without saddles on compact oriented surfaces

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Let M be a smooth compact and oriented surface, and denote by P a real line $\mathbb{R}$ or a circle S^1. Denote by $\mathcal{F}^0(M,P)$ a class of Morse-Bott functions without saddles on M with the value in P. This class of functions naturally arises in the study of homotopy type of stabilizers of Morse-Bott functions on surfaces with respect to the action of the group of diffeomorphisms by pre-composition, see details in [1]. It is known that this class is non-empty if M is diffeomorphic to one of the following list: a cylinder $S^1 \times [0,1]$, a disk D^2, a sphere S^2, a torus T^2.

There are some trivial examples of functions from $\mathcal{F}^0(M,P)$ that are easy to write by hand:

Example 1. Let $f_0 : M_0 \to P$ be a smooth function from $\mathcal{F}^0$

(1_0) $M_0 = S^1 \times [0,1] = \{(z,s) \,|\, z \in \mathbb{C}, |z| = 1,\, 0 \le s \le 1\}$ is a unit cylinder, and $f_0 : S^1 \times [0,1] \to \mathbb{R}$ is given by $f_0(z,s) = s$,

(2_0) $M_0 = D^2 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \le 1\}$ is a unit 2-disk, and $f_0 : D^2 \to \mathbb{R}$ is given by $f_0(x,y) = x^2 + y^2$,

(3_0) $M_0 = S^2 = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$ is a unit sphere, and $f_0 : S^2 \to \mathbb{R}$ is given by $f_0(x,y,z) = z$,

(4_0) $M_0 = T^2 = \{(w,z) \in \mathbb{C}^2 \,|\, |z| = |w| = 1\}$ is a unit 2-torus, and $f_0 : T^2 \to S^1$ is given by $f_0(w,z) = z$.

Note that these functions do not have critical circles. We will call them **prime functions**.

Our main result is the following theorem, see [2].

Theorem 2. *A function $f \in \mathcal{F}^0(M,P)$ admits the following decomposition*

$$f = \varkappa \circ f_0 \circ h^{-1} \tag{1}$$

where $h : M_0 \to M$ is a diffeomorphism, $f_0 \in \mathcal{F}^0(M_0,P)$ is a prime function, and a smooth function $\varkappa : f_0(M_0) \to P$ which satisfies the following conditions:

(A) *$\varkappa$ has the only finite number of non-degenerated critical points,*

(B) *$\varkappa$ does not have critical points at $f_0(\Sigma_{f_0})$ and $f_0(\partial M)$,*

where Σ_{f_0} is the set of critical points of f_0. A factorization (1) is not unique and depends on the choice of h. In particular, if f has no critical circles, then $\varkappa$ is a diffeomorphism.

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Szegő and Poisson kernels on Grauert Tubes and Lie group actions

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Let (M, κ) be a compact, connected, real-analytic Riemannian manifold. It is well known that M can be complexified in an essentially unique way, and that on a tubular neighborhood of M inside the complexification, there exists a Kähler structure compatible with the metric κ, with Kähler potential given by a real-analytic, strictly plurisubharmonic, and positive exhaustion function ρ. The sublevel sets $\rho < \tau^2$ are known as open Grauert tubes of radius τ, and they are strictly pseudoconvex domains in the complexification. Their boundaries, denoted X^τ, inherit a CR-holomorphic structure of codimension 1 and a natural contact structure.

If (M, κ) is endowed with an isometric action of a compact, connected Lie group G, this action lifts to a holomorphic action on the open Grauert tube and to a CR action on the boundary X^τ, both commuting with the Hamiltonian flow of ρ, known as the geodesic flow. These lifted actions give rise to unitary representations, respectively, on the eigenspaces of the Laplacian on (M, κ) and on the eigenspaces of an elliptic self-adjoint Toeplitz operator induced by the generator of the homogeneous geodesic flow on the the Hardy space $H(X^\tau)$.

This talk, based on joint work with R. Paoletti, has a twofold aim: first, to describe the scaling asymptotics of the equivariant Poisson-wave kernel, which relates to the asymptotic concentration of complexified eigenfunctions of the Laplacian in a fixed isotype, when restricted to X^τ; and second, to describe the scaling asymptotics of the equivariant Szegő kernel, which pertains to the asymptotic concentration of the eigenfunctions of the aforementioned Toeplitz operator in a given isotypical component.

Centenary of Quantum Theory: What Comes Next?

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This talk is dedicated to the centenary of the creation of quantum theory and on the occasion of the International Year of Quantum Science and Technology.

It will outline the chronology of the last 350 years of the theory of fundamental evolution equations, which represent the laws of Nature. The discussion will cover the origins of the evolution equations that describe quantum systems. Additionally, the talk will survey the mathematical structure of modern quantum theory and the prospects for its future development.

Examples of applications of fundamental evolution equations that have sparked the second quantum revolution in our time will also be provided.

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Weighted inequalities for sub-monotone functionals and applications

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In this talk, we will briefly discuss the history of weighted Hardy inequalities, and present new elementary and universal proof of the well known two weighted Hardy inequality from resent paper [1]. In the second part we present (see [2]) a set of relations between several quite diverse types of weighted inequalities involving various integral operators and fairly general quasi-norm-like functionals, which we call sub-monotone. The main result enables one to solve a specific problem by transferring it to another one for which a solution is known. Inequalities for Hardy, Copson, geometric mean and harmonic mean operators are shown to be interlinked. We give applications weighted inequalities restricted to cones of monotone functions [3].

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On Collectively AM, b-AM, KB, quasi-KB sets of operators

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A linearly ordered real vector space E is called a vector lattice if $sup\{x, y\}$ exists in E for every $x, y \in E$. Let E be a vector lattice. For each $x, y \in E$ with $x \leq y$, the set $[x, y] = \{z \in E : x \leq z \leq y\}$ is said to be an order interval. A subset A of E is called order bounded if it is included in some order interval. By E^+ we denote the set of all positive elements in E. A Banach space $(E, \|.\|)$ is called a Banach lattice if E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$.

Definition 1. A net (x_α) in an Archimedean vector lattice E is called order convergent to $x \in E$ if there exists a net (y_β) satisfying $y_\beta \downarrow 0$, and for any β there exists α_β such that $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_\beta$.

Definition 2. An operator T from a vector lattice E into a Banach lattice F is called AM-compact if the image of each order bounded subset of E is relatively compact in F.

Every regular compact operator is an AM-compact. The identity operator $I : l_1 \to l_1$ is an AM-compact operator, but it is not compact operator.

Definition 3. Let $\tau \subset L(X, Y)$. τ is called collectively AM-compact if for every order interval $[x, y]$ in X the set $\tau[x, y] = \bigcup_{T \in \tau} T[x, y]$ is relatively compact.

Definition 4. Let A and B be subsets of $L^+(X, Y)$. A is called dominated by B if, for each $S \in A$ there exists $T \in B$ such that $S \leq T$.

Definition 5. Let E, F be normed lattices and $T : E \to F$ is called KB-operator for every bounded increasing sequence (x_n) in E^+, there is an $x \in E$ such that (Tx_n) converges to Tx in norm.

Definition 6. Let E, F be normed lattices and $\tau \subseteq L(E, F)$. We say that τ is a collectively KB set of operators if, for every increasing bounded sequence (x_n) in E^+, there is an indexed subset $\{x_T\}_{T \in \tau}$ of E satisfying $\{(Tx_n) : T \in \tau\}$ norm converges to $\{Tx_T\}_{T \in \tau}$.

A collectively AM-compact($b - AM$ compact, KB)- set of operators is a generalization of the AM-compact($b - AM$ compact,KB) and quasi KB-operator. We investigate collective versions of some operators such as AM-compact, $b - AM$ compact, KB and quasi-KB operators.

We discuss the domination problem for collectively AM-compact,$b - AM$ compact, KB and quasi-KB sets of operators.

For this subject, we give the following references.

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On one geometric application of the Sturm–Hurwitz theorem

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For a smooth closed curve γ with curvatures $k_1 > 0$, $k_2 > 0$, and k_3 in the four-dimensional Euclidean space $\mathbb{E}^4$, we explore the well-defined integral quantity

$$J(\gamma) = \oint_\gamma \sqrt{k_1^2 + k_2^2 + k_3^2} ds,$$

which is invariant under rigid motions and dilatations in $\mathbb{E}^4$. We address the problem of determining the sharp lower bound for $J(\gamma)$, see [1].

Clearly, $J(\gamma) \geq 2\pi$ in view of the classical Fenchel inequality $\oint_\gamma k_1 ds \geq 2\pi$. However, if γ has constant curvatures then the stronger estimate $J(\gamma) \geq 2\sqrt{5}\pi$ holds true, and this estimate is sharp, see [2].

We conjecture that the same inequality $J(\gamma) \geq 2\sqrt{5}\pi$ holds true in the general situation as well. At the moment, the conjecture remains still unproven.

We consider the limit situation where γ evolves smoothly into a unit circle. Specifically, we introduce a smooth family of closed curves $\{\gamma_\varepsilon\}_{\varepsilon \geq 0}$ in $\mathbb{E}^4$ represented by the position vector $x(t) = (\cos t, \sin t, \varepsilon w_1(t), \varepsilon w_2(t))$, where $w_1(t)$, $w_2(t)$ are smooth 2π-periodic functions. This family is viewed as a perturbation of the unit circle γ_0.

Clearly, all the geometric features of γ_ε are determined by the vector-function $w(t) = (w_1(t), w_2(t))$. In particular, γ_ε with $\varepsilon > 0$ satisfy $k_1 > 0$ and $k_2 > 0$ if and only if $w(t)$ satisfies $w'' + w \neq 0$. In this generic case, the value of $J(\gamma_\varepsilon)$ is well-defined for $\varepsilon > 0$, and one can explore its limit value as $\varepsilon \to 0$.

We provide a geometrically meaningful description for the value of $\lim_{\varepsilon \to 0} J(\gamma_\varepsilon)$ in terms of the planar curve Γ represented by $p = w'' + w$, and then we demonstrate, as the main result, that this limit value cannot be less that $2\sqrt{5}\pi$,

$$\lim_{\varepsilon \to 0} J(\gamma_\varepsilon) \geq 2\sqrt{5}\pi,$$

for any choice of $w(t)$. Moreover, the inequality is proved to be sharp in the sense that one can chose $w(t)$ with $w'' + w \neq 0$ so that $\lim_{\varepsilon \to 0} J(\gamma_\varepsilon) = 2\sqrt{5}\pi$. Thus, the proved statement provides novel non-trivial arguments supporting the conjecture under consideration.

The proof of the main result is based on the use of the Sturm–Hurwitz theorem regarding the number of zeroes of trigonometric polynomials / Fourier series, see [3], [4]. We apply this celebrated theorem of the mathematical analysis to estimate a specific tangency complexity of the planar curve Γ leading to the desired lower bound for $\lim_{\varepsilon \to 0} J(\gamma_\varepsilon)$.

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Harmonic number series related to specific generating functions

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The central binomial coefficients are defined for integer $n \geq 0$ by $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$. These numbers are closely related to the well-known Catalan numbers, given by $C_n = \frac{1}{n+1}\binom{2n}{n}$. The harmonic numbers of order m and the odd harmonic numbers of order m are defined respectively by $H_n^{(m)} = \sum\limits_{k=1}^{n} \frac{1}{k^m}$ and $O_n^{(m)} = \sum\limits_{k=1}^{n} \frac{1}{(2k-1)^m}$ with $H_0^{(m)} = O_0^{(m)} = 0$. The cases $H_n^{(1)} = H_n$ and $O_n^{(1)} = O_n$ correspond to the ordinary harmonic and odd harmonic numbers, respectively.

In this note, we present several infinite series involving central binomial coefficients, Catalan numbers, harmonic numbers, products of harmonic numbers, and mixed products with odd harmonic numbers. The method used to derive these expressions relies on integration—a classical technique that has recently gained renewed attention in the literature [1–3], among others.

Theorem 1. *We have*

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} \frac{H_n}{n} = \frac{\pi^2}{3}, \qquad \sum_{n=0}^{\infty} \frac{C_n H_{n+1}}{2^{2n}} = 4,$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} \frac{H_{n+r}}{n+r} = \frac{2^{2r+1}}{\binom{2r}{r}} \frac{O_r}{r}, \quad r \neq 0,$$

and more generally, for $s - \frac{1}{2} \notin \mathbb{Z}_{<0}$, $r \neq 0$, *and* $r + s - \frac{1}{2} \notin \mathbb{Z}_{<0}$,

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} \frac{H_{n+r+s} - H_s}{\binom{n+r+s}{s+1}} = \frac{2(s+1)}{2s+1} \frac{H_{r+s-1/2} - H_{s-1/2}}{\binom{r+s-1/2}{r-1}}.$$

Theorem 2. *We have*

$$\sum_{n=0}^{\infty} \frac{O_{n+1}}{(n+1)(2n+1)} = \frac{\pi^2}{6}, \qquad \sum_{n=0}^{\infty} \frac{C_n O_{n+3}}{2^{2n}(2n+5)} = \frac{8}{9} + \frac{\pi}{32} - \frac{\pi \ln 2}{4},$$

$$\sum_{n=0}^{\infty} \frac{C_n H_{n+r+1}}{2^{2n+1}(n+r+1)} = \frac{H_r}{r} - \frac{2^{2r}}{4r^2-1} \frac{O_{r+1}}{\binom{2(r-1)}{r-1}},$$

and more generally, for $s, r \in \mathbb{C} \setminus \mathbb{Z}_{\le 0}$, *and* $r + s - \frac{1}{2} \notin \mathbb{Z}_{<0}$,

$$\sum_{n=0}^{\infty} \frac{C_n}{2^{2n+1}} \frac{H_{n+r+s} - H_{s-1}}{\binom{n+r+s}{s}} = \frac{H_{r+s-1} - H_{s-1}}{\binom{r+s-1}{s}} - \frac{2s}{2s+1} \frac{H_{r+s-1/2} - H_{s-1/2}}{\binom{r+s-1/2}{r-1}}.$$

Theorem 3. *We have*

$$\sum_{n=1}^{\infty} \frac{H_n H_{n+1}}{n\,(n+1)} = \frac{\pi^2}{6} + 2\zeta(3), \qquad \sum_{n=1}^{\infty} \frac{H_n O_n}{(2n-3)(2n-1)} = \frac{\pi^2}{36} - \frac{1}{6} + \frac{1}{2} \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{H_n H_{n+s+1}}{(n+s)(n+s+1)} = \frac{H_s}{s} + \frac{H_s^2 + H_s^{(2)}}{s}, \quad s \ne 0,$$

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $(\Re(s) > 1)$ *is the Riemann zeta function. More generally, for* $0 \le r \in \mathbb{C} \setminus \mathbb{Z}_{<0}$, $s \in \mathbb{C} \setminus \mathbb{Z}_{\le 0}$,

$$\sum_{n=1}^{\infty} \frac{H_n H_{n+r+s+1}}{\binom{n+r+s+1}{r+2}} = \frac{r+2}{s\binom{r+s}{r}} H_{r+1}(H_{r+s} - H_r) + \frac{r+2}{s\binom{r+s}{r}} \Big((H_{r+s} - H_r)^2 + H_{r+s}^{(2)} - H_r^{(2)} \Big).$$

Theorem 4. *We have*

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} \frac{O_n}{n} = \sum_{n=1}^{\infty} \frac{2^{2n} H_n}{n(n+1)\binom{2(n+1)}{n+1}}, \qquad \sum_{n=1}^{\infty} \frac{C_n O_n}{2^{2n}} = \sum_{n=1}^{\infty} \frac{2^{2(n+1)} H_n}{(n+1)(n+2)\binom{2(n+2)}{n+2}},$$

and more generally, if $r \in \mathbb{Z}_{\ge 0}$, *then*

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n} O_n}{2^{2n}\,(n+r)} = \sum_{n=1}^{\infty} \frac{2^{2(n+r)} H_n}{(n+r)(n+r+1)\binom{2(n+r+1)}{n+r+1}}.$$

Theorem 5. *For all* $x \in [-1/4, 1/4)$,

$$2\sqrt{1-4x} \sum_{n=1}^{\infty} \binom{2n}{n} H_n O_n x^n$$
$$= \frac{\pi^2}{2} + \mathrm{Li}_2(1-4x) - 4\mathrm{Li}_2\big(\sqrt{1-4x}\big) + \ln(1-4x) \left(\ln\left(\frac{1-4x}{|x|}\right) - \frac{1 - \operatorname{sgn} x}{2}\, \pi i \right),$$

where $\mathrm{Li}_2(x)$ *denotes the dilogarithm function, defined by* $\mathrm{Li}_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}, |x| \le 1$.

Theorem 6. *We have*

$$\sum_{n=1}^{\infty} \frac{C_n H_n O_n}{2^{2n}} = \frac{\pi^2}{2} + 4\ln 2, \qquad \sum_{n=1}^{\infty} \frac{C_n H_n O_n}{2^{2n}\,(n+2)} = -\frac{14}{9} + \frac{\pi^2}{6} + \frac{4}{9} \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{H_n O_n}{(2n+1)\,(2n+3)} = \frac{1}{2} + \frac{\pi^2}{24} - \frac{1}{2} \ln 2,$$

and, more generally, for $r \in \mathbb{C} \setminus \mathbb{Z}_{\le 0}$,

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{\binom{n+(r+1)/2}{n}} \frac{H_n O_n}{2^{2n}} = \frac{2(r+1)}{r^2} \left(H_r - H_{r/2} - \frac{r}{4} H_{r/2}^{(2)} + \frac{\pi^2}{24} r - \ln 2 + \frac{2}{r} \right).$$

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On non-topologizable semigroups of the bicyclic monoid

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In this paper we shall follow the semigroup terminology of [1, 2, 4, 5].

Throughout these abstract we always assume that all topological spaces involved are Hausdorff — unless explicitly stated otherwise.

Definition 1. Let X, Y and Z be topological spaces. A map $f\colon X\times Y\to Z$, $(x,y)\mapsto f(x,y)$, is called

- (i) *right* [*left*] *continuous* if it is continuous in the right [left] variable; i.e., for every fixed $x_0\in X$ [$y_0\in Y$] the map $Y\to Z$, $y\mapsto f(x_0,y)$ [$X\to Z$, $x\mapsto f(x,y_0)$] is continuous;
- (ii) *separately continuous* if it is both left and right continuous;
- (iii) *jointly continuous* if it is continuous as a map between the product space $X\times Y$ and the space Z.

Definition 2. Let S be a non-void topological space which is provided with an associative multiplication (a semigroup operation) $\mu\colon S\times S\to S$, $(x,y)\mapsto\mu(x,y)=xy$. Then the pair (S,μ) is called

- (i) a *right topological semigroup* if the map μ is right continuous, i.e., all interior left shifts $\lambda_s\colon S\to S$, $x\mapsto sx$, are continuous maps, $s\in S$;
- (ii) a *left topological semigroup* if the map μ is left continuous, i.e., all interior right shifts $\rho_s\colon S\to S$, $x\mapsto xs$, are continuous maps, $s\in S$;
- (iii) a *semitopological semigroup* if the map μ is separately continuous;
- (iv) a *topological semigroup* if the map μ is jointly continuous.

We usually omit the reference to μ and write simply S instead of (S,μ). It goes without saying that every topological semigroup is also semitopological and every semitopological semigroup is both a right and left topological semigroup.

A topology τ on a semigroup S is called:

- a *semigroup* topology if (S,τ) is a topological semigroup;
- a *shift-continuous* topology if (S,τ) is a semitopological semigroup;
- an *left-continuous* topology if (S,τ) is a left topological semigroup;
- an *right-continuous* topology if (S,τ) is a right topological semigroup.

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq=1$. The semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$q^k p^l \cdot q^m p^n = \begin{cases} q^{k-l+m} p^n, & \text{if } l < m; \\ q^k p^n, & \text{if } l = m; \\ q^k p^{l-m+n}, & \text{if } l > m. \end{cases}$$

We define the following subsets of the bicyclic monoid

$$\mathscr{C}_+(p,q) = \left\{ q^i p^j \in \mathscr{C}(p,q) \colon i \leqslant j \right\} \quad \text{and} \quad \mathscr{C}_-(p,q) = \left\{ q^i p^j \in \mathscr{C}(p,q) \colon i \geqslant j \right\}.$$

For an arbitrary non-negative integer k we define

$$\mathscr{C}_{+k}(a,b) = \left\{ b^i a^{i+s} \in \mathscr{C}_+(a,b) \colon s \geqslant k, s \in \omega \right\}.$$

Fix an arbitrary infinite subset X of ω. Latter we shall assume that $X = \{x_i \colon i \in \omega\}$ where $\{x_i\}_{i\in\omega}$ is a steadily increasing sequence in ω. Put

$$\mathscr{C}_{+k}^X(a,b) = \mathscr{C}_{+k}(a,b) \cup \{ b^{x_i} a^{x_i} \in \mathscr{C}_+(a,b) \colon i \in \omega \}.$$

The set $\mathscr{C}_{+k}^X(a,b)$ is a subsemigroup of $\mathscr{C}_+(a,b)$. By dual way we define the subsemigroup $\mathscr{C}_{-k}^X(a,b)$ of $\mathscr{C}_-(a,b)$.

Theorem 3. *The monoid $\mathscr{C}_+(a,b)$ contains continuum many non-isomorphic subsemigroups of the forms $\mathscr{C}_{+k}^X(a,b)$, where k is a positive integer and X is an infinite subset of ω, such that every left-continuous Hausdorff topology on $\mathscr{C}_{+k}^X(a,b)$ is discrete.*

Theorem 4. *The monoid $\mathscr{C}_-(a,b)$ contains continuum many non-isomorphic subsemigroups of the forms $\mathscr{C}_{-k}^X(a,b)$, where k is a positive integer and X is an infinite subset of ω, such that every right-continuous Hausdorff topology on $\mathscr{C}_{+k}^X(a,b)$ is discrete.*

Proposition 5. *The monoid $\mathscr{C}_+(a,b)$ contains continuum many non-isomorphic subsemigroups of the forms $\mathscr{C}_{+k}^X(a,b)$, where k is a positive integer and X is an infinite subset of ω, and there exists a Hausdorff topology τ on $\mathscr{C}_{+k}^X(a,b)$ such that the semigroup operation on $(\mathscr{C}_{+k}^X(a,b),\tau)$ is left-continuous but it is not right-continuous.*

Proposition 6. *The monoid $\mathscr{C}_-(a,b)$ contains continuum many non-isomorphic subsemigroups of the forms $\mathscr{C}_{-k}^X(a,b)$, where k is a positive integer and X is an infinite subset of ω, and there exists a Hausdorff topology τ on $\mathscr{C}_{-k}^X(a,b)$ such that the semigroup operation on $(\mathscr{C}_{-k}^X(a,b),\tau)$ the semigroup operation is left-continuous but it is not right-continuous.*

The set $\mathscr{C}_{\mathbb{Z}} = \mathbb{Z} \times \mathbb{Z}$ with the following semigroup operation

$$(k,l) \cdot (m,n) = \begin{cases} (k-l+m, n), & \text{if } l < m; \\ (k,n), & \text{if } l = m; \\ (k, l-m+n), & \text{if } l > m. \end{cases}$$

is called the *extended bicyclic semigroup* [6]. Every Hausdorff shift-continuous topology on the semigroup $\mathscr{C}_{\mathbb{Z}}$ is discrete [3]. We construct continuum subsemigroups S of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ such that the statements of Theorem 3 and Propositions 5 (Theorem 4 and Propositions 6) hold for S and every element of S is not maximal with the respect to the induced natural partial order from the inverse semigroup $\mathscr{C}_{\mathbb{Z}}$.

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Stability of minimal surfaces in three-dimensional sub-Riemannian Lie groups

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A *sub-Riemannian manifold* is a smooth manifold M together with a completely non-integrable smooth distribution $\mathcal{H}$ on M (it is called a *horizontal distribution*) and a smooth field of Euclidean scalar products $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ on $\mathcal{H}$ (it is called a *sub-Riemannian metric*). In particular, $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ can be constructed as a restriction of some Riemannian metrics $\langle\cdot,\cdot\rangle$ on M to $\mathcal{H}$. Here we will assume that all sub-Riemannian structures are of this form. Let Σ be a smooth oriented surface in a three-dimensional sub-Riemannian manifold M. If N_h is the orthogonal projection of the unit normal field N of Σ (in the Riemannian sense) onto $\mathcal{H}$ and $d\Sigma$ is the Riemannian area form of Σ, then the *sub-Riemannian area* of a domain $D \subset \Sigma$ is defined as $A(D) = \int_D |N_h|\, d\Sigma$. The *normal variation* of the surface Σ defined by a smooth function u is the map $\varphi\colon \Sigma \times I \to M : \varphi_s(p) = \exp_p(su(p)N(p))$, where I is an open neighborhood of 0 in $\mathbb{R}$ and $\exp_p$ is the Riemannian exponential map in p. Denote $A(s) = \int_{\Sigma_s} |N_h|\, d\Sigma_s$, where $\Sigma_s = \varphi_s(\Sigma)$. Then $A'(0)$ is called the *first (normal) area variation* defined by φ, and $A''(0)$ is called the *second* one. A surface Σ is called *minimal* if $A'(0) = 0$ for any normal variations with compact support in $\Sigma \setminus \Sigma_0$, where $\Sigma_0 = \{p \in \Sigma \mid N_h(p) = 0\}$ is the *singular set* of Σ. A minimal surface Σ is called *stable* if $A''(0) \geq 0$ for any normal variations with compact support in $\Sigma \setminus \Sigma_0$. We will call a surface Σ in a three-dimensional sub-Riemannian manifold *vertical* if $T_p\Sigma \perp \mathcal{H}_p$ for each $p \in \Sigma$. In particular, for such surfaces $N_h = N$ and $\Sigma_0 = \varnothing$.

In [1] we proved that a vertical surface Σ is minimal in the sub-Riemannian sense if and only if it is minimal in the Riemannian sense and derived the following second variation formula:

$$A''(0) = \int_\Sigma -\left(X(u) - \langle\nabla_N X, N\rangle u\right)^2 + |\nabla_\Sigma u|^2 - \left(\mathrm{Ric}\,(N,N) + |B|^2\right) u^2\, d\Sigma,$$

where ∇ and Ric are the Riemannian connection and the Ricci tensor of M respectively, X is the unit normal vector field of $\mathcal{H}$ (which is tangent to Σ because it is vertical), ∇_Σ and B are the Riemannian gradient and the second fundamental form of Σ respectively. It follows that if Σ is stable in the sub-Riemannian sense, it is also stable in the Riemannian sense.

The three-dimensional Riemannian Heisenberg group (also known as the three-dimensional Thurston geometry Nil) is the space $\mathbb{R}^3$ with coordinates (x, y, z) and with the following orthonormal basis of left-invariant vector fields defined by its nilpotent Lie group structure:

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z},\ X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z},\ X_3 = \frac{\partial}{\partial z}.$$

Theorem 1. *Let a sub-Riemannian structure on Nil be defined by a left-invariant two-dimensional horizontal distribution. Then its normal field should be of the form* $X = \frac{1}{\sqrt{\lambda^2+\mu^2+1}}(\lambda X_1 + \mu X_2 + X_3)$.

If $\lambda = \mu = 0$ then a complete connected vertical surface in this sub-Riemannian manifold is minimal if and only if it is a vertical Euclidean plane. In the other case it is minimal if and only if it is a vertical Euclidean plane over a straight line in the (x, y)-plane that has the direction (λ, μ).

All these surfaces are stable in the sub-Riemannian sense and thus in the Riemannian sense.

The three-dimensional Thurston geometry *Sol* is the space $\mathbb{R}^3$ with coordinates (x, y, z) and with the following orthonormal basis of left-invariant vector fields defined by its solvable Lie group structure:

$$X_1 = e^{-z}\frac{\partial}{\partial x},\ X_2 = e^{z}\frac{\partial}{\partial y},\ X_3 = \frac{\partial}{\partial z}.$$

Theorem 2. *Let a sub-Riemannian structure on Sol be defined by a left-invariant two-dimensional horizontal distribution. Then its normal field should be of the form $X = \frac{1}{\sqrt{\lambda^2+\mu^2+\nu^2}}(\lambda X_1 + \mu X_2 + \nu X_3)$, where $\lambda\mu \neq 0$.*

If $\nu \neq 0$ then a complete connected vertical surface in this sub-Riemannian manifold is minimal if and only if it is cylindrical and can be parameterized either as

$$r(s,t) = \left(x_0 - \frac{\lambda}{\nu}e^{-s}, t, s\right) \quad \text{or as} \quad r(s,t) = \left(t, y_0 + \frac{\mu}{\nu}e^{s}, s\right).$$

If $\nu = 0$ then a complete connected vertical surface is minimal if and only if it is a horizontal Euclidean plane $z = z_0$ or $\lambda = \pm\mu$ and the surface is a "hyperbolic helicoid" (previously described in [2]) with the parameterization

$$r(s,t) = \left(x_0 + \frac{1}{\sqrt{2}}e^{-t}s, y_0 \pm \frac{1}{\sqrt{2}}e^{t}s, t\right).$$

All these surfaces are stable in the sub-Riemannian sense and thus in the Riemannian sense.

The three-dimensional Thurston geometry $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ is the universal covering of the special linear group $\mathrm{SL}(2,\mathbb{R})$. It also can be described as the universal covering of the unit tangent bundle of the hyperbolic plane $\mathbb{H}^2$ with the Sasaki metric. Thus, using the half-plane model of $\mathbb{H}^2$, we can present $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ as the half-space $\{(x,y,z) \in \mathbb{R}^3 \mid y > 0\}$ with the orthonormal frame

$$Y_1 = y\frac{\partial}{\partial x} - \frac{\partial}{\partial z},\ Y_2 = y\frac{\partial}{\partial x},\ Y_3 = \frac{\partial}{\partial z}.$$

Note that the fields Y_1 and Y_2 here are not left-invariant.

Theorem 3. *A two-dimensional horizontal distribution $\mathcal{H} = X^{\perp}$, whose normal field X is a linear combination of the fields Y_1–Y_3 with constant coefficients, defines a sub-Riemannian structure on $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ (i.e., is its horizontal distribution) if and only if X is of the form $\frac{1}{\sqrt{\lambda^2+\mu^2+1}}(\lambda Y_1 + \mu Y_2 + Y_3)$, where $\lambda \neq -1$. This sub-Riemannian structure allows vertical minimal surfaces only for $\lambda = 0$ and $\lambda = 1$.*

If $\mu \neq 0$ then a complete connected vertical surface is minimal if and only if it is a half-plane $x = x_0$ for $\lambda = 0$ or a half-plane $z = z_0$ for $\lambda = 1$.

If $\mu = 0$ and $\lambda = 1$ then a complete connected vertical surface is minimal if and only if it is either a half-plane $z = z_0$ or can be parameterized as

$$r(s,t) = \left(y_0 s\cos t, y_0\cos t, \sqrt{2}t + z_0\right).$$

If $\mu = \lambda = 0$ then a complete connected vertical surface in this sub-Riemannian manifold is minimal if and only if is a cylinder over a geodesic in $\mathbb{H}^2$ (see, e.g., [3]).

All these surfaces are stable in the sub-Riemannian sense and thus in the Riemannian sense.

We also find vertical minimal surfaces of a left-invariant sub-Riemannian structure defined by a horizontal distribution $\mathcal{H} = X^{\perp}$, where $X = y\cos z\frac{\partial}{\partial x} + y\sin z\frac{\partial}{\partial y} - \cos z\frac{\partial}{\partial z}$, and establish their stability.

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Rational factorization of Lax type flows in the space dual to the centrally extended Lie algebra of fractional integro-differential operators

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The Lie-algebraic approach to the rational factorization of Lax type flows in spaces dual to certain operator Lie algebras (see, for example, [1]) and central extensions of some of them is developed for the central extension of the Lie algebra $\mathbb{A}_\alpha := \mathbb{A}_0\{\{D^\alpha, D^{-\alpha}\}\}$, consisting of fractional integro-differential operators such as $\mathrm{a}_\alpha := \sum_{j\in\mathbb{Z}_+} a_j D^{\alpha(p_\alpha - j)}$, where $\mathbb{A}_0 := A\{\{D, D^{-1}\}\}$ is the Lie algebra of ordinary integral-differential operators, $A := W_2^\infty(\mathbb{R};\mathbb{C})\cap\, W_\infty^\infty(\mathbb{R};\mathbb{C})$, $D^\alpha : A \to A$ is a Riemann-Liouville fractional derivative, $\alpha \in \mathbb{C}\setminus\mathbb{Z}$, $\operatorname{Re}\alpha \neq 0$, $a_j \in \mathbb{A}_0$, $j \in \mathbb{Z}_+$, $p_\alpha \in \mathbb{N}$ is an order of the fractional integro-differential operator a_α. This Lie algebra possesses the standard commutator $[\mathrm{a}_\alpha, \mathrm{b}_\alpha] = \mathrm{a}_\alpha \circ \mathrm{b}_\alpha - \mathrm{b}_\alpha \circ \mathrm{a}_\alpha$, and invariant with respect to this commutator scalar product $(\mathrm{a}_\alpha, \mathrm{b}_\alpha) := \int_{\mathbb{R}} \operatorname{res}_D (\operatorname{res}_{D_\alpha} (\mathrm{a}_\alpha \circ \mathrm{b}_\alpha D^{-\alpha}))dx$, where $\mathrm{a}_\alpha, \mathrm{b}_\alpha \in \mathbb{A}_\alpha$, "$\circ$" is a symbol of the operator product, $\operatorname{res}_{D_\alpha}$ denotes a coefficient at $D^{-\alpha}$ for any fractional integral-differential operator as well as res_D denotes a coefficient at D^{-1} for any ordinary integral-differential operator.

On the central extension $\hat{\mathbb{A}}_\alpha := \bar{\mathbb{A}}_\alpha \oplus \mathbb{C}$ of the parameterized Lie algebra $\bar{\mathbb{A}}_\alpha := \prod_{y\in\mathbb{S}^1} \mathbb{A}_\alpha$ by the Maurer-Cartan 2-cocycle $\omega_2(\mathrm{a}_\alpha, \mathrm{b}_\alpha) := \langle \mathrm{a}_\alpha, \partial \mathrm{b}_\alpha/\partial y\rangle$, where $\langle \mathrm{a}_\alpha, \mathrm{b}_\alpha\rangle = \int_{\mathbb{S}^1} (\mathrm{a}_\alpha, \mathrm{b}_\alpha)dy$, $\mathrm{a}_\alpha, \mathrm{b}_\alpha \in \bar{\mathbb{A}}_\alpha$, there exist the commutator

$$[(\mathrm{a}_\alpha, d), (\mathrm{b}_\alpha, e)] = ([\mathrm{a}_\alpha, \mathrm{b}_\alpha], \omega_2(\mathrm{a}_\alpha, \mathrm{b}_\alpha)), \qquad (\mathrm{a}_\alpha, d), (\mathrm{b}_\alpha, e) \in \hat{\mathbb{A}}_\alpha,$$

and corresponding invariant scalar product

$$((\mathrm{a}_\alpha, d), (\mathrm{b}_\alpha, e)) = \langle \mathrm{a}_\alpha, \mathrm{b}_\alpha\rangle + ed. \tag{1}$$

The Lie algebra $\bar{\mathbb{A}}_\alpha$ allows the splitting into the direct sum of its two Lie subalgebras $\bar{\mathbb{A}}_\alpha = \bar{\mathbb{A}}_{\alpha,+} \oplus \bar{\mathbb{A}}_{\alpha,-}$, where $\bar{\mathbb{A}}_{\alpha,+}$ is the Lie subalgebra of the formal power series by the operator D^α. On the space $\hat{\mathbb{A}}_\alpha^*$ dual to the central extension $\hat{\mathbb{A}}_\alpha$ with respect to the scalar product (1) the $\mathcal{R}$-deformed commutator

$$[(\mathrm{a}_\alpha, d), (\mathrm{b}_\alpha, e)]_{\mathcal{R}} = ([\mathrm{a}_\alpha, \mathrm{b}_\alpha]_{\mathcal{R}}, \omega_{2,\mathcal{R}}(\mathrm{a}_\alpha, \mathrm{b}_\alpha)),$$
$$[\mathrm{a}_\alpha, \mathrm{b}_\alpha]_{\mathcal{R}} = [\mathcal{R}\mathrm{a}_\alpha, \mathrm{b}_\alpha] + [\mathrm{a}_\alpha, \mathcal{R}\mathrm{b}_\alpha], \qquad \omega_{2,\mathcal{R}}(\mathrm{a}_\alpha, \mathrm{b}_\alpha) = \omega_2(\mathcal{R}\mathrm{a}_\alpha, \mathrm{b}_\alpha) + \omega_2(\mathrm{a}_\alpha, \mathcal{R}\mathrm{b}_\alpha),$$

where $\mathcal{R}:\bar{\mathbb{A}}_\alpha\to\bar{\mathbb{A}}_\alpha$ is a space endomorphism, $\mathcal{R}=(P_+-P_-)/2$, $P_\pm$ are projectors on $\mathbb{A}_{\alpha,\pm}$ accordingly, generates the Lie-Poisson bracket

$$\{\gamma,\mu\}_{\mathcal{R}}(l_\alpha,c)=\langle l_\alpha,[\nabla\gamma(l_\alpha),\nabla\mu(l_\alpha)]_{\mathcal{R}}\rangle+c\omega_{2,\mathcal{R}}(\nabla\gamma(l_\alpha),\nabla\mu(l_\alpha)):=\langle\nabla\gamma(l_\alpha),\Theta\nabla\mu(l_\alpha)\rangle, \quad (2)$$

where $\gamma,\mu\in\mathcal{D}(\bar{\mathbb{A}}_\alpha^*)$ are smooth by Frechet functionals on $\bar{\mathbb{A}}_\alpha^*\simeq\bar{\mathbb{A}}_\alpha$, "$\nabla$" is a symbol of the functional gradient, the Poisson operator $\Theta:T^*(\bar{\mathbb{A}}_\alpha^*)\to T(\bar{\mathbb{A}}_\alpha^*)$ acts by the rule

$$\Theta:\nabla\gamma(l_\alpha)\mapsto-[l_\alpha-c\partial/\partial y,(\nabla\gamma(l_\alpha))_-]+[l_\alpha-c\partial/\partial y,\nabla\gamma(l_\alpha)]_{\leq 0}$$

for any smooth by Frechet functional $\gamma\in\mathcal{D}(\bar{\mathbb{A}}_\alpha^*)$, $T(\bar{\mathbb{A}}_\alpha^*)$ and $T^*(\bar{\mathbb{A}}_\alpha^*)$ are tangent and cotangent spaces to $\bar{\mathbb{A}}_\alpha^*$ accordingly, $(l_\alpha,c)\in\hat{\mathbb{A}}_\alpha^*\simeq\hat{\mathbb{A}}_\alpha$, $l_\alpha\in\bar{\mathbb{A}}_\alpha^*$ is a fractional integro-differential operator of the order $q_\alpha\in\mathbb{N}$. By means of the Casimir invariants $\gamma_n\in I(\hat{\mathbb{A}}_\alpha^*)$, $n\in\mathbb{N}$, satisfying the relationship $[l_\alpha-c\partial/\partial y,\nabla\gamma_n(l_\alpha)]=0$ at a point $(l_\alpha,c)\in\hat{\mathbb{A}}_\alpha^*$, as Hamiltonians, in the dual space $\hat{\mathbb{A}}_\alpha^*\simeq\hat{\mathbb{A}}_\alpha$ the Lie-Poisson bracket (2) determimes the hierarchy of Lax type Hamiltonian flows in the form

$$\partial l_\alpha/\partial t_n=[(\nabla\gamma_n(l_\alpha))_+,l_\alpha-c\partial/\partial y],\qquad t_n\in\mathbb{R},\quad n\in\mathbb{N}, \quad (3)$$

where the subscript "+" denotes the projection of the corresponding element from $\bar{\mathbb{A}}_\alpha$ on the Lie subalgebra $\bar{\mathbb{A}}_{\alpha,+}$ and $\nabla\gamma_n(l_\alpha):=\sum_{j\in\mathbb{Z}_+}a_{n,j}D^{\alpha(n-j)}$. One considers another hierarchy of Lax type Hamiltonian flows on the dual space $\hat{\mathbb{A}}_\alpha^*$ such as

$$dl_\alpha/dt_n=[(\nabla\gamma_n(l_\alpha))_+,\tilde{l}_\alpha-c\partial/\partial y],\qquad t_n\in\mathbb{R},\quad n\in\mathbb{N}, \quad (4)$$

for some fractional integro-differential operator $\tilde{l}\in\bar{\mathbb{A}}_\alpha^*$ of the order $q_\alpha\in\mathbb{N}$, which is related with the operator $l_\alpha\in\bar{\mathbb{A}}_\alpha^*$ by the generalized gauge transformation

$$\tilde{l}_\alpha(0)-c\partial/\partial y=B_\alpha(0)^{-1}(l_\alpha(0)-c\partial/\partial y)B_\alpha(0), \quad (5)$$

where $B_\alpha(0)\in\bar{\mathbb{A}}_{\alpha,+}$ is some fractional differential operator of the order $s_\alpha\in\mathbb{N}$ with constant coefficients, at the initial moment of the time $t_n\in\mathbb{R}$ for every $n\in\mathbb{N}$.

Theorem 1. *If for every $n\in\mathbb{N}$ at the initial moment of the time $t_n\in\mathbb{R}$ the fractional integro-differential operators $l_\alpha,\tilde{l}_\alpha\in\tilde{\mathrm{g}}^*$ of the order $q_\alpha\in\mathbb{N}$ are related by the relationship (5), there exist such fractional differential operators $A_\alpha,B_\alpha\in\bar{\mathbb{A}}_{\alpha,+}$ of the orders $q_\alpha+s_\alpha$ and s_α accordingly, where $s_\alpha\in\mathbb{Z}_+$, $s_\alpha<q_\alpha$, that the equalities*

$$l_\alpha=A_\alpha B_\alpha^{-1},\quad \tilde{l}_\alpha=B_\alpha^{-1}(A_\alpha-c\partial B_\alpha/\partial y) \quad (6)$$

hold. The operators $A_\alpha,B_\alpha\in\bar{\mathbb{A}}_{\alpha,+}$ satisfy the following systems of evolution equations

$$\begin{aligned}dA_\alpha/dt_n&=(\nabla\gamma_n(l_\alpha))_+A_\alpha-A_\alpha(\nabla\gamma_n(\tilde{l}))_++c(\partial(\nabla\gamma_n(l_\alpha))_+/\partial y)B_\alpha,\\ dB_\alpha/dt_n&=(\nabla\gamma_n(l_\alpha))_+B_\alpha-B_\alpha(\nabla\gamma_n(\tilde{l}_\alpha))_+,\quad n\in\mathbb{N},\end{aligned} \quad (7)$$

or, equivalently,

$$\begin{aligned}dA_\alpha/dt_n&=(A_\alpha(\nabla\gamma_n(l_\alpha))_-)_+-((\nabla\gamma_n(\tilde{l}_\alpha))_-A_\alpha)_+-c((\partial(\nabla\gamma_n(l_\alpha))_-/\partial y)B_\alpha)_+,\\ dB_\alpha/dt_n&=(B_\alpha(\nabla\gamma_n(l_\alpha))_-)_+-((\nabla\gamma_n(\tilde{l}_\alpha))_-B_\alpha)_+,\qquad n\in\mathbb{N}.\end{aligned}$$

which possess an infinite sequence of the conservation laws $H_n\in\mathcal{D}(\bar{\mathbb{A}}_{\alpha,+}\times\bar{\mathbb{A}}_{\alpha,+})$, $n\in\mathbb{N}$, in the forms

$$H_n(A_\alpha,B_\alpha):=\gamma_n(l_\alpha)|_{l_\alpha=A_\alpha B_\alpha^{-1}}=\gamma_n(\tilde{l}_\alpha)|_{\tilde{l}_\alpha=B_\alpha^{-1}(A_\alpha-c\partial B_\alpha/\partial y)}.$$

Theorem 2. *For every $n\in\mathbb{N}$ the system of evolution equations (7), given on the subspace $\bar{\mathbb{A}}_{\alpha,+}\times\bar{\mathbb{A}}_{\alpha,+}\subset\bar{\mathbb{A}}_\alpha\times\bar{\mathbb{A}}_\alpha$, is Hamiltonian with respect to the Poisson bracket $\{.,.\}_{\mathcal{L}}$ which arises as a reduction of the Poisson bracket $\{.,.\}_{\bar{\mathcal{L}}}$ with the corresponding Poisson operator $\bar{\mathcal{L}}=(P')^{-1}(\Theta\oplus\tilde{\Theta})(P'^*)^{-1}$, where $\tilde{\Theta}$ is a Poisson operator generating the Poisson bracket (2) at a point $\tilde{l}\in\bar{\mathbb{A}}_\alpha^*$,*

$P'^{*}: T^{*}(\bar{\mathbb{A}}_{\alpha}^{*} \oplus \bar{\mathbb{A}}_{\alpha}^{*}) \to T^{*}(\bar{\mathbb{A}}_{\alpha} \times \bar{\mathbb{A}}_{\alpha})$ is an operator adjoint to the Frechet derivative $P': T(\bar{\mathbb{A}}_{\alpha} \times \bar{\mathbb{A}}_{\alpha}) \to T(\bar{\mathbb{A}}_{\alpha}^{*} \oplus \bar{\mathbb{A}}_{\alpha}^{*})$ of the Backlund transformation $P: (A_{\alpha}, B_{\alpha}) \in \bar{\mathbb{A}}_{\alpha} \times \bar{\mathbb{A}}_{\alpha} \mapsto (l_{\alpha}, \tilde{l}_{\alpha}) \in \bar{\mathbb{A}}_{\alpha}^{*} \oplus \bar{\mathbb{A}}_{\alpha}^{*}$, determined by the equalities (6), $(P'^{*})^{-1}$ is inverse to one, on $\bar{\mathbb{A}}_{\alpha,+} \times \bar{\mathbb{A}}_{\alpha,+}$ and Hamiltonians $\bar{H}_{n} \in \mathcal{D}(\bar{\mathbb{A}}_{\alpha,+} \times \bar{\mathbb{A}}_{\alpha,+})$ in the forms

$$\bar{H}_{n}(A_{\alpha}, B_{\alpha}) := \gamma_{n}(l_{\alpha})|_{l_{\alpha}=A_{\alpha}B_{\alpha}^{-1}} + \gamma_{n}(\tilde{l}_{\alpha})|_{\tilde{l}_{\alpha}=B_{\alpha}^{-1}(A_{\alpha}-c\partial B_{\alpha}/\partial y)}, \qquad n \in \mathbb{N}.$$

In the case of $c = 0$ the second Hamiltonian representation for the hierarchy (7) is also found.

The rational factorization method for the central extension $\hat{\mathbb{A}}_{\alpha}$ is applied to construct a new integrable hierarchy of two-dimensional nonlinear dynamical systems with fractional derivatives by one spatial variable as well as a new integrable hierarchy of two-dimensional hydrodynamic Benney-type systems, which is its quasiclassical approximation.

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Criteria of optimality of some classes of simple functions on surfaces with the boundary

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Smooth functions are the tool of research in many scientific fields. Their classification and optimality problems are important enough. There are a number of papers dedicated to functions with non-degenerated critical points on the boundary of a surface [1, 3, 4, 5, 7, 8] and with inner [2, 10] and boundary [6, 9] isolated critical points on the low-dimensional manifolds.

A function is *optimal* if it has the smallest number of critical points among all functions on present surface (if such exists). Also the function, which has no more than one critical point on each level line, is called a *simple function*.

A simple Morse function being defined on a surface with the boundary, is called a *mm-function*, if its restriction to the boundary is also a Morse function and all critical points belong to the boundary of the surface.

In this thesis we have presented the criterias of optimality of the following classes of simple functions: (1) mm-functions on a surface with the boundary; (2) Morse functions on a closed oriented connected surface; (3) functions with isolated critical points on the boundary of a connected surface with the connected boundary.

Theorem 1. *A mm-function being defined on a surface of genus g with k components of the boundary is optimal if and only if it has $4g + 2k$ critical points if the surface is oriented and $2g + 2k$ critical points if the surface is non-oriented.*

A function is polar if it has exactly one minimum and one maximum point on a present manifold.

Theorem 2. *A Morse function on a closed oriented surface is optimal i and only if it is polar on a present surface.*

Theorem 3. *Mm-function on a smooth compact oriented surface with the boundary is optimal if and only if it is polar on a present surface.*

Let M be a smooth compact oriented surface with the connected boundary ∂M and let $f : M \to \mathbb{R}$ be a smooth function defined on M with finitely many critical points on the boundary. Remark that the number of critical points is finite is equivalent to their isolatedness. Let $CP(f)$ $(ICP(f))$ be a set of (isolated) critical points of the function f and f_∂ be a restriction of the function f to the boundary ∂M of the surface M. Then we are going to consider the following set of functions:

$$\Theta(M) = \{f : M \to \mathbb{R} | f \in C^\infty(M), CP(f) = ICP(f) = ICP(f_\partial), f \text{ is simple}\}$$

Theorem 4. *Suppose that $f \in \Theta(M)$ and M is a connected compact surface with connected boundary, which is not homeomorphic to a two-dimensional disk. Then the function f is optimal if and only if it has exactly three critical points. Then the function f is optimal if and only if it has exactly two critical points in the case of two-dimension disk.*

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The approximate solution of the Boltzmann equation for the hard sphere model

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The Boltzmann kinetic equation plays an important role in the kinetic theory of gases. It is describes the evolution of rarefied gases. For the hard sphere model, the equation has the form [1]

$$D(f) = Q(f, f); \tag{1}$$

$$D(f) \equiv \frac{\partial f}{\partial t} + \left(V, \frac{\partial f}{\partial x}\right), \tag{2}$$

$$Q(f, f) \equiv \frac{d^2}{2} \int_{R^3} dV_1 \int_{\Sigma} d\alpha |(V - V_1, \alpha)| \Big[f(t, x, V_1') f(t, x, V') - f(t, x, V) f(t, x, V_1) \Big]. \tag{3}$$

The only exact solution to equation (1), which is known explicitly up to now, is the Maxwell distribution M or simply Maxwellian (after J. C. Maxwell, Scottish physicist). It makes both parts of the Boltzmann equation equal to zero, namely

$$D(M) = 0, \quad Q(M, M) = 0. \tag{4}$$

The solution to this equation (1)-(3) will be look for in the next form[2]

$$f(t, x, V) = \sum_{i=1}^{\infty} \varphi_i(t, x) M_i(t, x, V), \tag{5}$$

where coefficient functions $\varphi_i(t, x)$ are nonnegative smooth functions on $\mathbb{R}^4$. M_i are Maxwellians (4), which describe the eddy-like motion of the gas.

As a measure of the deviation between the parts of equation (1) we will consider a uniform-integral error of the form:

$$\Delta = \sup_{(t,x) \in \mathbb{R}^4} \int_{\mathbb{R}^3} \big| D(f) - Q(f, f) \big| dV. \tag{6}$$

In the paper [2], we were obtained sufficient conditions for the coefficient functions and hydrodynamic parameters appearing in the distribution, which enable one to make the analyzed error (6) as small as desired.

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Dynamics of operators on the space of Radon measures

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In this talk, we will consider the dynamics of the adjoint of a weighted composition operator and we will give necessary and sufficient conditions for this adjoint operator to be topologically transitive on the space of Radon measures on a locally compact Hausdorff space. Moreover, we will provide sufficient conditions for this operator to be chaotic and we will give concrete examples. Next, we will consider the real Banach space of signed Radon measures and we will give in this context sufficient conditions for the convergence of Markov chains induced by the adjoint of an integral operator. Also, we will illustrate this result by a concrete example. In addition, we will present some structural results regarding the space of Radon measures. More precisely, we will characterize a class of cones whose complement is spaceable in the space of Radon measures.

The talk will be based on [1].

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Analysis of Weak Associativity in Some Hyper-Algebraic Structures that Represent Dismutation Reactions

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In this paper, some chemical systems of Tin (Sn), Indium (In) and Vanadium (V) which are represented by hyper-algebraic structures $(S_{Sn}, \oplus)$, $(S_{In}, \oplus)$ and $(S_V, \oplus)$ were studied. The analyses of their algebraic properties and the probabilities of elements in dismutation reactions were carried out with the aid of computer codes in Python programming language. It was shown that in the dismutation reactions, the left nuclear (N_λ)-probability, middle nuclear (N_μ)-probability and right nuclear (N_ρ)-probability for each of the hyper-algebraic structures $(S_{Sn}, \oplus)$, $(S_{In}, \oplus)$ and $(S_V, \oplus)$ is less than 1.000. This implies that, $(S_{Sn}, \oplus)$, $(S_{In}, \oplus)$ and $(S_V, \oplus)$ are non-associative hyper-algebraic structures. Also, from the results obtained for FLEX-probability, it was shown that, $(S_{Sn}, \oplus)$, $(S_{In}, \oplus)$ and $(S_V, \oplus)$ have flexible elements because the values of their FLEX-probabilities are 1.000 each. Hence, $(S_{Sn}, \oplus)$, $(S_{In}, \oplus)$ and $(S_V, \oplus)$ are flexible. Overall, $(S_V, \oplus)$ exhibited the lowest measure of weak-associativity, $(S_{Sn}, \oplus)$ exhibited lower measure of weak-associativity, and $(S_{In}, \oplus)$ exhibited a low measure of weak-associativity.

Definition 1. (Semihypergroup, Quasihypergroup, Hypergroup, H_v-group)

An hypergroupoid or polygroupoid $(H, \circ)$ is the pair of a non-empty set H with an hyperoperation $\circ : H \times H \to P(H)\backslash\{\emptyset\}$ defined on it.

An hypergroupoid $(H, \circ)$ is called a semihypergroup if

(i): it obeys the associativity law $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in H$, which means that

$$\bigcup_{u \in\ a \circ b} u \circ c = \bigcup_{v \in\ b \circ c} a \circ v$$

An hypergroupoid $(H, \circ)$ is called a quasihypergroup if

(ii): it obeys the reproduction axiom $x \circ H = H = H \circ x$ for all $x \in H$.

An hypergroupoid $(H, \circ)$ is called an hypergroup if it is a semihypergroup and a quasihypergroup.

A hypergroupoid $(H, \circ)$ is called an H_v-semigroup it obeys the weak associativity (WASS) condition

(iii): $x \circ (y \circ z) \cap (x \circ y) \circ z \neq \emptyset$ for all $x, y, z \in H$.

A hypergroupoid $(H, \circ)$ is called an H_v-group if it is a quasihypergroup and a H_v-semigroup.

According to Davvaz et al. [1], all combinational probabilities for the set $S_{Sn} = \{Sn, Sn^{2+}, Sn^{4+}\}$ without energy can be displayed as follows in Table 1.1 and the major products are shown. $(S_{Sn}, \oplus)$

Table 1.1. Dismutation reaction for tin (Sn)

$\oplus$	Sn	Sn^{2+}	Sn^{4+}
Sn	Sn	Sn, Sn^{2+}	Sn^{2+}
Sn^{2+}	Sn, Sn^{2+}	Sn^{2+}	Sn^{2+}, Sn^{4+}
Sn^{4+}	Sn^{2+}	Sn^{2+}, Sn^{4+}	Sn^{4+}

is not a quasihypergroup, not a semihypergroup, not a hypergroup and not an H_v-group. But it is an H_v-semigroup. Summarily, even though $(S_{Sn}, \oplus)$ is not associative and has weak associativity, it is commutative. However, it has an hyper-substructure that is associative, i.e. an hypergroup.

Definition 2. (Left nuclear element)

Let $(P, \cdot)$ be a polygroupoid. The left nucleus pair of $x \in P$ will be denoted by $N_\lambda(x)$ and defined as $N_\lambda(x) = \{(y, z) \in P \times P \mid x \cdot (yz) = (xy) \cdot z\}$. $x \in P$ will be said to be left nuclear if $N_\lambda(x) = P \times P$.

Definition 3. (Probability of left nuclear element/polygroupoid)

Let $(P, \cdot)$ be a polygroupoid.

(1) The probability of an element $x \in P$ being left nuclear will be denoted by $Pr_{N_\lambda(P,\cdot)}(x)$ and will be defined as $Pr_{N_\lambda(P,\cdot)}(x) = \dfrac{|N_\lambda(x)|}{|P|^2}$.

(2) The probability of $(P, \cdot)$ being left nuclear will be denoted by $Pr_{N_\lambda}(P, \cdot)$ and defined as $Pr_{N_\lambda}(P, \cdot) = \dfrac{\sum_{x \in P} Pr_{N_\lambda(P,\cdot)}(x)}{|P|}$.

Based on our Definition 2 and Definition 3, we have the result below.

Lemma 4. *Let $(P, \cdot)$ be a polygroupoid. Let the left nucleus of $(P, \cdot)$ be defined as $N_\lambda(P, \cdot) = \{x \in P \mid x \cdot (yz) = (xy) \cdot z \ \forall \ (y, z) \in P \times P\}$. Then:*

(1) $N_\lambda(P, \cdot) = \{x \in P \mid N_\lambda(x) = P \times P\} = \{x \in P \mid x \text{ is left nuclear}\}$.

(2) $$Pr_{N_\lambda}(P, \cdot) = \frac{\sum_{x \in P} |N_\lambda(x)|}{|P|^3}.$$

Using Lemma 4, the results in Table 4.2 were gotten using the information in Table 1.1.

Table 4.2. Probability of elements in dismutation reaction tin, S_{Sn}

Probability of Properties	Sn	Sn^{2+}	Sn^{4+}	S_{Sn}
Left Nucleus $P_{N_\lambda}(\cdot)$	0.5556	0.7778	0.5556	0.6297
Middle Nucleus $P_{N_\mu}(\cdot)$	0.5556	0.7778	0.5556	0.6297
Right Nucleus $P_{N_\rho}(\cdot)$	0.5556	0.7778	0.5556	0.6297
Flexibility $P_{FLEX}(\cdot)$	1.0000	1.0000	1.0000	1.0000
Left Alternative Property $P_{LAP}(\cdot)$	0.6667	1.0000	0.6667	0.7778
Right Alternative Property $P_{RAP}(\cdot)$	0.6667	1.0000	0.6667	0.7778

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On special affinor structures on the Gaudi's surface

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We investigated the existence of a special affinor structure on the surface, which is named after the Catalan architect Antonio Gaudí.

The Gaudí's surface is given in R_3 by the general equation $z = kxsin(\frac{y}{a})$, were k, a - some constants, or by the parametric equations

$$x = u^1, \quad y = u^2, \quad z = ku^1 sin(\frac{u^2}{a}).$$

As is known [2] an affinor structure $F_i^h(x)$ in Riemannian space $(V_n(x), g_{ij}(x))$ that satisfies condition

$$F_\alpha^h F_i^\alpha = e\delta_i^h, \qquad e = 0, \pm 1, \tag{1}$$
$$i, h, j, ... = 1, 2, ...n,$$

is called

- *elliptic* if $e=-1$,
- *hyperbolic* if $e=+1$,
- *m-parabolic* when $e=0$, $rankF=m\ \ (2m<n)$,
- *parabolic* when $e=0$, $rankF=m\ \ (2m=n)$.

Usually the affinor structure F_i^h is coordinated with the Riemannian metric g_{ij} as follows

$$g_{i\alpha}F_j^{\alpha}=-g_{j\alpha}F_i^{\alpha}. \qquad (2)$$

So, we are looking for an affinor structure $F_i^h(u)$ on the Gaudi's surface $(V_2^G(u),g_{ij}(u))$, $i,j,h=1,2$ provided that $a=k=1$, that satisfies conditions (1), (2). Then

$$(g_{ij}(u))=\begin{pmatrix} 1+Sin^2u^2 & u^1Cosu^2Sinu^2 \\ u^1Cosu^2Sinu^2 & 1+(u^1)^2Cos^2u^2 \end{pmatrix},$$

As a result, it turned out that the Gaudi's surface does not admit an affinor e-structure of hyperbolic and parabolic types, but it admits an elliptic affinor structure

$$(F_i^h(u))=\begin{pmatrix} \frac{-u^1Cosu^2Sinu^2}{1+sin^2u^2+(u^1)^2Cos^2u^2} & \frac{-1-(u^1)^2Cos^2u^2}{1+sin^2u^2+(u^1)^2Cos^2u^2} \\ \frac{1+Sin^2u^2}{1+sin^2u^2+(u^1)^2Cos^2u^2} & \frac{u^1Cosu^2Sinu^2}{1+sin^2u^2+(u^1)^2Cos^2u^2} \end{pmatrix},$$

which is necessarily absolutely parallel:

$$F_{i,j}^h=0.$$

Here comma «,» is a sign of the covariant derivative in respect to the connection of V_2^G, that is the Gaudi's surface admits a Kähler structure [1].

In this case, the corresponding fundamental 2-form has the form

$$(F_{ij}(u))=(g_{i\alpha}(u)F_j^{\alpha}(u))=\begin{pmatrix} 0 & -(1+sin^2u^2+(u^1)^2Cos^2u^2)^{0,5} \\ (1+sin^2u^2+(u^1)^2Cos^2u^2)^{0,5} & 0 \end{pmatrix}.$$

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Reconstructing Morse functions with prescribed preimages of single points

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Morse functions have been fundamental and strong tools in geometry. Morse functions are also important and interesting objects in geometry. For Morse functions and handles, which are fundamental tools and objects, see [5] for example.

We consider the following fundamental problem. This was essentially started in [1].

Problem 1. Let $m>1$ be an integer and $a_1<a_2$ real numbers. Let F_1 and F_2 be smooth closed manifolds of dimension $m-1$. Can we reconstruct a Morse function $\tilde{f}_{a_1,a_2}:\tilde{M}_{a_1,a_2}\to\mathbb{R}$ on some m-dimensional compact and connected manifold $\tilde{M}_{a_1,a_2}$ onto the closed interval $[a_1,a_2]$ enjoying the following?

(1) The boundary of $\tilde{M}_{a_1,a_2}$ is diffeomorphic to $F_1 \sqcup F_2$ and the preimage $\tilde{f}_{a_1,a_2}^{-1}(a_i)$ is diffeomorphic to F_i.
(2) There exists a unique critical value a and a is in the open interval (a_1, a_2). The preimage $\tilde{f}_{a_1,a_2}^{-1}(a)$ is connected.

In the case $m = 2$, F_i must be a disjoint union of circles and related studies had been presented before. For this, Sharko [8] first considered the reconstruction of smooth functions with critical points being represented by some elementary polynomials, on closed surfaces. Later, Michalak [6] has explicitly solved Problem 1 in the case $m = 2$ (and $F_i = \sqcup S^{m-1}$ where S^{m-1} is the $(m-1)$-dimensional unit sphere). There, he has also classified Morse functions on given closed surfaces in terms of their *Reeb graphs*: the *Reeb graph* of a smooth function $c : X \to \mathbb{R}$ on a manifold X with no boundary is the quotient space W_c of the manifold obtained by the equivalence relation regarding that two points are equivalent if and only if they are in a same connected component of a same preimage $c^{-1}(y)$. They are classical objects and have already appeared in [7]. They have been fundamental and strong tools in understanding the manifolds compactly.

We present our study and result.

Definition 2 ([3])**.** A *most fundamental handlebody* of dimension m is a smooth, compact and connected manifold diffeomorphic to one obtained by attaching finitely many handles to the boundary S^{m-1} of the m-dimensional unit disk D^m disjointly and simultaneously where at least one handle is attached.

Example 3. The unit disk D^m and a compact manifold represented as a boundary connected sum $\natural_j(S^{k_j} \times D^{m-k_j})$ $(1 \leq k_j \leq m-1)$ (considered in the smooth category) are m-dimensional most fundamental handlebodies.

Theorem 4 ([1, 2])**.** *In the case each of connected components of F_1 and F_2 is the boundary of some most fundamental handlebody of dimension m, Problem 1 is affirmatively solved.*

A main ingredient of the proof is as follows: by attaching handles to $F_1 \times \{1\} \subset F_1 \times [0,1]$ disjointly, simultaneously and suitably, we have an m-dimensional smooth compact and connected manifold $\tilde{M}_{a_1,a_2}$ whose boundary is diffeomorphic to $F_1 \sqcup F_2$. Note that [1] also shows local functions around local extrema which belong to a certain class generalizing the class of Morse(-Bott) functions, as another result.

Corollary 5 ([3])**.** *In the case $m = 4$ with F_j $(j = 1, 2)$ being orientable, Problem 1 is affirmatively solved.*

This comes from fundamental and important facts on 3-dimensional manifold theory.

We review elementary properties of closed surfaces and introduce some elementary numerical invariants. A closed and connected surface F is diffeomorphic to a connected sum of the form $(\sharp_{j_1=1}^{k_1}(S^1 \times S^1))\sharp(\sharp_{j_2=1}^{k_2}(\mathbb{R}P^2))$ where k_1 and k_2 are non-negative integers. A closed and connected surface F is orientable if and only if $k_2 = 0$. We can define $P(F) = k_2$ as a topological invariant for closed and connected surfaces and we can extend this to closed surfaces F which may not be connected in the additive way. Note that if $P(F)$ is odd, then this is not the boundary of any 3-dimensional compact manifold. We can define another topological invariant $P_o(F)$ for closed surfaces F as the numbers of connected components F_j of F with $P(F_j)$ being odd.

Theorem 6 ([2, 4])**.** *Problem 1 is solved affirmatively in the case $m = 3$ if and only if either the following three hold.*

(1) $P_o(F_1) = P_o(F_2)$.

(2) $P_o(F_1) - P_o(F_2)$ *is even,* $P_o(F_1) > P_o(F_2)$*, and* $P_o(F_1) \leq P(F_2)$*.*
(3) $P_o(F_1) - P_o(F_2)$ *is even,* $P_o(F_1) < P_o(F_2)$*, and* $P(F_1) \geq P_o(F_2)$*.*

For this, the case F_i are orientable is a specific case of Theorem 4. The condition has been shown to be sufficient in [2] first by explicit construction of Morse functions. [4] has shown that the condition is also a necessary condition by investigating attachment of handles to $F_1 \times \{1\} \subset F_1 \times [0,1]$ to have a smooth, compact and connected manifold $\tilde{M}_{a_1,a_2}$ whose boundary is diffeomorphic to $F_1 \sqcup F_2$, precisely. [4] is also a kind of addenda to [2].

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Umbilics on complete convex planes : The Toponogov Conjecture

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In 1995, Victor Andreevich Toponogov [1] authored the following conjecture: “Every smooth non-compact strictly convex and complete surface of genus zero has an umbilic point, possibly at infinity“. In our talk, we will outline the 2024 proof in collaboration with Brendan Guilfoyle [2].

Theorem 1. [2] *Assume that* $P \hookrightarrow \mathbb{R}^3$ *be a proper embedded strictly convex surface, and assume that it is diffeomorphic to the plane and* $C^{3,\alpha}$ *- regular. Then*

$$\inf_{p \in P} |\kappa_1(p) - \kappa_2(p)| = 0.$$

Namely we prove that, should a counter-example to the Conjecture exist, (a) the Fredholm index of an associated Riemann Hilbert boundary problem for holomorphic discs is negative [3]. Thereby, (b) no such holomorphic discs survive for a generic perturbation of the boundary condition (these form a Banach manifold under the assumption that the Conjecture is incorrect). Finally, however, (c) the geometrization by a neutral Kaehler metric [4] of the associated model allows for Mean Curvature Flow [5] with mixed Dirichlet – Neumann boundary conditions to generate a holomorphic disc from an initial spacelike disc. This completes the indirect proof of said conjecture as (b) and (c) are in contradiction. Note that our regularity assumption is stronger than the minimal required in the context, which would be twice continuously differentiable.

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The p-adic class numbers of Z-covers of graphs

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In the spirit of arithmetic topology, we propose to study the p-adic limit values of the number of spanning trees in pro-p covers of graphs. This talk is based on a joint work [KU25] and will focus on a specific example.

Let X be a *finite connected graph*, that is, a 1-dimensional CW complex. A *spanning tree* T of X is a connected subgraph that contains all vertices and no loops. The number of spanning trees of each X is denoted by $k(X)$. A basic reference for graphs is [Ter11].

Suppose that X is *the 8-graph*, consisting of one vertex and two looped edges. Let s_1, s_2 denote the elements of the fundamental group $\pi_1(X)$ represented by the two loops. We consider a specific surjective homomorphism

$$\varphi : \pi_1(X) \to \mathbb{Z};\ s_1 \mapsto 1, s_2 \mapsto 2.$$

The $\mathbb{Z}$-cover $X_\infty \to X$ corrreponding to $\operatorname{Ker}\varphi$ is so-called *the Fibonacci tower*. The adjacency matrix yields *the Ihara zeta function* and *the Ihara polynomial* $I(t) = 4 - (t + 1/t) - (t^2 + 1/t^2)$. We further put $J(t) := t^2 I(t)/(t-1)^2 = t^2 + 3t + 1$.

For each $n \in \mathbb{Z}_{>0}$, let $X_n \to X$ denote the $\mathbb{Z}/n\mathbb{Z}$-subcover. Then, Pengo–Vallieres [PV25, Theorem 3.6] asserts that the number of spanning trees of X_n may be calculated by using the cyclic resultant $\mathrm{Res}(t^n-1,J(t))=\prod_{\zeta^n=1}J(\zeta)\in\mathbb{Z}$ as

$$k(X_n)=k(X)n^{2-1}|\mathrm{Res}(t^n-1,J(t))|/J(1).$$

On the other hand, p being a prime number, Kisilevsky [Kis97] and Ueki–Yoshizaki [UY25] proved that p-power-th cyclic resultant p-adically converges in the ring of p-adic integers $\mathbb{Z}_p=\varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ and gave explicit formulae. For instanse, if $\Phi_m(t)\in\mathbb{Z}[t]$ denote the m-th cyclotomic polynomial and $f(t)\in\mathbb{Z}[t]$ satisfies $f(t)\equiv\Phi_m(t) \bmod p$, then

$$\lim_{n\to\infty}\mathrm{Res}(t^{p^n}-1,f(t))=\Phi_m(1)$$

holds.

Combining the above, we obtain the following for the 8-graph X.

Theorem 1. *The sequence $(k(X_{p^n}))_n$ converges in $\mathbb{Z}_p$. We have*

$$\lim_{n\to\infty}k(X_{p^n})/p^n\in\mathbb{Q}\underset{\text{iff}}{\iff}p=2,3,5.$$

In addition, if we put $r_n:=|\mathrm{Res}(t^n-1,J(t))|$, then we have

$$\lim_{n\to\infty}|r_{2^n}|=-3,\ \lim_{n\to\infty}|r_{3^n}|=2,\ \lim_{n\to\infty}|r_{5^n}|=0.$$

The non-5 part of $|r_{5^n}|$ is $|r_{5^n}|/5^{2n+1}$. Fix an embedding $\overline{\mathbb{Q}}\hookrightarrow\widehat{\overline{\mathbb{Q}_5}}$ of an algebraic closure of $\mathbb{Q}$ into the completion of an algebraic closure of the 5-adic number field. Let α,β denote the roots of $J(t)$ and let log denote the 5-adic logarithm. Then we have

$$\lim_{n\to\infty}|r_{5^n}|/5^{2n+1}=\frac{\log\alpha\log\beta}{5}\in\mathbb{Z}_5.$$

We may observe that these sequences converge quickly:

If $p=2$,

n	1	2	3	4	5	6
$-\mathrm{Res}(t^{2^n}-1,J(t))$	5	45	2205	4870845	23725150497405	...
$-\mathrm{Res}(t^{2^n}-1,J(t)) \bmod 2^n$	-3	-3	-3	-3	-3	-3

If $p=3$,

n	1	2	3	4	5	6
$\mathrm{Res}(t^{3^n}-1,J(t))$	20	5780	192900153620	...	...	...
$\mathrm{Res}(t^{3^n}-1,J(t)) \bmod 3^n$	2	2	2	2	2	2

If $p=5$,

n	1	2	3	4	5	6
$\mathrm{Res}(t^{5^n}-1,J(t))$	5^3	$5^5\cdot3001^2$	$5^7\cdot3001^2\cdot158414167964045700001^2$	...	...	...
$\frac{1}{5^{2n+1}}\mathrm{Res}(t^{5^n}-1,J(t)) \bmod 5^n$	1	1	1	376	2876	15376

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2F-planar mappings of Riemannian spaces with a special affinor structure

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The concept of 2F-planar mapping ($2FPM$) of spaces with affine connection and Riemannian spaces was defined by R. J. Kadem [1]. These mappings are a natural generalization of geodesic [2] and F-planar mappings [3]. R. J. Kadem investigated general problems of $2FPM$ theory. In particular, he proved that every such mapping preserves affinor structure.

We study $2FPM$ of Riemannian spaces with a special type of f-structure

$$(V_n, g_{ij}, F_i^h) \longrightarrow (\overline{V}_n, \overline{g}_{ij}, F_i^h).$$

The fundamental equations of such a mapping in the common coordinate system (x^i) with respect to the $2FPM$ has the form:

$$\overline{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + \psi_{(i}(x)\delta_{j)}^h + \phi_{(i}(x)F_{j)}^h(x) + \sigma_{(i}(x)F_{|\alpha|}^h(x)F_{j)}^\alpha(x), \tag{1}$$

$$F_i^h(x) = \overline{F}_i^h(x),$$

$$g_{i\alpha}F_j^\alpha = -g_{j\alpha}F_i^\alpha, \qquad \overline{g}_{i\alpha}F_j^\alpha = -\overline{g}_{j\alpha}F_i^\alpha, \tag{2}$$

$$F_{(i,j)}^h = q_{(i}F_{j)}^h, \tag{3}$$

$$F_\alpha^h F_\beta^\alpha F_i^\beta + F_i^h = 0, \tag{4}$$

$$i, h, j, ... = 1, 2, ...n,$$

where $\Gamma_{ij}^h, \overline{\Gamma}_{ij}^h$ are the Christoffel symbols of $V_n, \overline{V}_n$, respectively; $\psi_i(x)$, $\phi_i(x)$, $\sigma_i(x)$, $q_i(x)$ are certain covectors; $F_i^h(x)$ is affinor; brackets (i, j) denote the symmetrization with respect to the corresponding indices; comma «,» is a sign of the covariant derivative in respect to the connection of V_n.

We call an affinor structure F_i^h that satisfies conditions (3) a *generalized-recurrent structure* and $q_i(x)$ - the *generalized-recurrent vector*.

We have obtained the properties of the Riemannian and Ricci tensors of the generalized recurrent f-space.

The relationship between vectors $\psi_i(x)$, $\phi_i(x)$, $\sigma_i(x)$, $q_i(x)$ under conditions (1), (2), (3), (4) was found.

It is proved that the class of generalized recurrent spaces (V_n, g_{ij}, F_i^h) is closed with respect to $2FPM$, that is the space $(\overline{V}_n, \overline{g}_{ij}, F_i^h)$ under conditions (1), (2), (3), (4) is also generalized recurrent, but the vector of generalized recurrence is generally not preserved:

$$F_{(i|j)}^h = \tilde{q}_{(i}F_{j)}^h, \tag{5}$$

were «|» is a sign of the covariant derivative in respect to the connection of $\overline{V}_n$ and

$$\tilde{q}_i = q_i - \psi_i + \phi_\alpha F_i^\alpha + \sigma_i.$$

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Braid Group Action on Homology and its applications

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Given a branched cover of the $\mathbb{CP}^1$ with a standard description of permutation representation through $\sigma_1 \cdots \sigma_r \in S_n$ such that the genus is 1 we give a computational criteria to answer a question when these branched covers are of full moduli dimension.

Theorem 1. *Given a cover represented by the permutation representation as above form a moduli space of such. If there is an element of the fundamental group of such space (which is the subgroup of the braid group) acts on the homology basis of such cover with an element of an infinite order then the cover is of the full moduli dimension (i.e. a generic curve of genus* 1 *carries a function with such permutation representation.)*

The second application of these ideas is to compute fundamental groups of complex surfaces that are covers of $(CP^2 - D_2)/SL_2(\mathbb{C})$.

Weyl algebras and generalized symmetries

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Weyl algebras are fundamental objects in ring theory that arise from various perspectives in mathematics and physics, and the development of their theory is related to such names as Dirac, Heisenberg, Littlewood, Weyl, Segal, Dixmier and Kashiwara. Their feature is capturing the noncommutativity of differential operators with polynomial coefficients, which makes these algebras ubiquitous in abstract algebra, noncommutative geometry, representation theory and quantum mechanics. The representation theory of Weyl algebras led to the development of the so-called algebraic analysis, an advanced branch of algebra within whose framework several long-standing conjectures have been proven.

Following [1, 3], let $\mathbb{K}$ be a field of characteristic zero. The first Weyl algebra A_1 is the associative algebra over $\mathbb{K}$ generated by elements x and ∂ that satisfy the defining relation $\partial x - x\partial = 1$. The Weyl algebra A_1 is a central, simple, Noetherian, hereditary domain of Gelfand–Kirillov dimension two which is canonically isomorphic to the ring of differential operators $\mathbb{K}[x][\frac{\mathrm{d}}{\mathrm{d}x}]$ with coefficients from the polynomial ring $\mathbb{K}[x]$. The Bergman's diamond lemma [2] allows one to easily show that the tuple $(x^k\partial^l \mid k,l \in \mathbb{N}_0)$ is a basis of A_1. The nth Weyl algebra A_n is the tensor product $A_1 \otimes \cdots \otimes A_1$ of n copies of the first Weyl algebra.

From the perspective of symmetry analysis of differential equations, the first and the second real Weyl algebras arise as the algebras of linear generalized symmetries of the linear (1+1)-dimensional heat equation $u_t = u_{xx}$ and of the remarkable (1+2)-dimensional Fokker–Planck equation $u_t + xu_y = u_{xx}$, see [5] and [8], respectively. The above is only one way the close relationship between these two equations manifests itself. This relationship was revealed in the course of extended symmetry analysis of the latter and former equations in [5, 6, 7] and [4, 8], but it can in fact be embedded in a broader framework.

For each $n \in \mathbb{N}$, consider the class $\mathcal{U}_n$ of (ultra)parabolic linear second-order partial differential equations with $1+n$ independent variables t, x_1, ..., x_n and dependent variable u, where the corresponding (symmetric) matrices of coefficients of second-order derivatives of the dependent variable u are of rank one, and the number $n+1$ of independent variables is essential: none among them plays the role of a parameter even up to their point transformations. The equation

$$\mathcal{F}_n\colon\quad u_t + x_1u_{x_2} + \cdots + x_{n-1}u_{x_n} = u_{x_1x_1}$$

belongs to the class $\mathcal{U}_n$. Notably, the equations $\mathcal{F}_1$ and $\mathcal{F}_2$ coincide with the above linear heat and remarkable Fokker–Planck equations, respectively. The classes $\mathcal{U}_1$ and $\mathcal{U}_2$ coincide with the classes of parabolic linear second-order partial differential equation with two independent variables and of ultraparabolic linear second-order partial differential equations with three independent variables, respectively.

In this talk, we present the results of our in-depth preliminary analysis of the properties of the equations $\mathcal{F}_n$ within their respective classes $\mathcal{U}_n$. Among many surprising observations and conjectures, there are the following:

- The dimension of the essential Lie invariance algebra $\mathfrak{g}_n^{\rm ess}$ of $\mathcal{F}_n$ is equal to $2n+4$, and this algebra is isomorphic to the algebra ${\rm sl}(2,\mathbb{R}) \in_{\rho_{2n-1}\oplus\rho_0} {\rm h}(n,\mathbb{R})$. The Levi factor $\mathfrak{f}_n$ and the (nil)radical $\mathfrak{r}_n$ of $\mathfrak{g}_n^{\rm ess}$ are isomorphic to the real degree-two special linear algebra ${\rm sl}(2,\mathbb{R})$ and the rank-n Heisenberg algebra ${\rm h}(n,\mathbb{R})$, respectively. Here ρ_m denotes the standard real irreducible representation of ${\rm sl}(2,\mathbb{R})$ in the $(m+1)$-dimensional vector space.
- The dimension of $\mathfrak{g}_n^{\rm ess}$ is maximal among those of the essential Lie invariance algebras of equations from the class $\mathcal{U}_n$, and each equation whose essential Lie invariance algebra is of this maximal dimension is reduced to $\mathcal{F}_n$ by a point transformation in the space $\mathbb{R}^{1+n}_{t,x_1,\dots,x_n} \times \mathbb{R}_u$.
- The essential point-symmetry group $G_n^{\rm ess}$ of the equation $\mathcal{F}_n$ is isomorphic to the Lie group $\big({\rm SL}(2,\mathbb{R}) \ltimes_{\varrho_{2n-1}\oplus\varrho_0} {\rm H}(n,\mathbb{R})\big) \times \mathbb{Z}_2$, where ${\rm H}(n,\mathbb{R})$ denotes the rank-n Heisenberg group and ϱ_m is the irreducible representation of the real degree-two special linear group ${\rm SL}(2,\mathbb{R})$ in $\mathbb{R}^{m+1}$.
- A complete list of discrete point symmetry transformations of the equation $\mathcal{F}_n$ that are independent up to combining with each other and with continuous point symmetry transformations of this equation is exhausted by the single involution I alternating the sign of u, $I\colon (t,x_1,\dots,x_n,u) \mapsto (t,x_1,\dots,x_n,-u)$. Thus, the quotient group of the complete point-symmetry pseudogroup G_n of $\mathcal{F}_n$ with respect to its identity component is isomorphic to $\mathbb{Z}_2$.

- The algebra of canonical representatives of generalized symmetries of $\mathcal{F}_n$ is $\Sigma_n = \Lambda_n \Subset \Sigma_n^{-\infty}$. Here Λ_n is the subalgebra of linear generalized symmetries of $\mathcal{F}_n$, which is generated by acting with the Lie-symmetry operators associated with the canonical basis of the complement of the center $\langle u\partial_u\rangle$ in the (nil)radical $\mathfrak{r}_n$ of $\mathfrak{g}_n^{\rm ess}$ on the elementary seed symmetry vector field $u\partial_u$, and $\Sigma_n^{-\infty}$ is the ideal associated with linear superposition of solutions of $\mathcal{F}_n$.
- The algebra Λ_n is isomorphic to the Lie algebra $A_n^{(-)}$ associated with the nth Weyl algebra A_n.
- A generalized vector field is a master symmetry of $\mathcal{F}_n$ in the sense of the definition given in [7, p. 315] if and only if up to a triviality equivalence relation, it is a generalized symmetry of $\mathcal{F}_n$.
- The algebra Λ_n is two-generated as a Lie algebra, i.e., there is a pair of its elements such that Λ_n coincides with its subalgebra containing all successive commutators (aka nonassociative monomials) of these two elements.

This work introduces a substantial research program aimed at a deeper understanding of the symmetry properties of linear second-order partial differential equations.

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On the prime ends extension of unclosed inverse mappings

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The following statements contain itself some results on prime end boundary extension of quasiconformal mappings.

Theorem A. *Under a quasiconformal mapping f of a collared domain D_0 onto a domain D, there exists a one-to-one correspondence between the boundary points of D_0 and the prime ends of D. Moreover, the cluster set $C(f,b)$, $b\in\partial D_0$, coincides with the impression $I(P)$ of the corresponding prime end P of D (see* [1, Theorem 4.1]*).*

Given $f: D\rightarrow D'$, we set $C(f,x):=\{y\in\overline{\mathbb{R}^n}: \exists\, x_k\in D: x_k\rightarrow x, f(x_k)\rightarrow y, k\rightarrow\infty\}$ and $C(f,\partial D)=\bigcup\limits_{x\in\partial D}C(f,x)\,.$

Theorem B. *Let $f:D\rightarrow\mathbb{R}^n$ be quasiregular mapping with $C(f,\partial D)\subset\partial f(D)$. If D is locally connected at a point $b\in\partial D$ and $D'=f(D)$ is qc accessible at some point $y\in C(f,b)$, then $C(f,b)=\{y\}$ (see* [2, Theorem 4.2]*).*

The goal of this abstract is to consider mappings which are not closed. Let $y_0\in\mathbb{R}^n$, $0<r_1<r_2<\infty$ and $A=A(y_0,r_1,r_2)=\{y\in\mathbb{R}^n: r_1<|y-y_0|<r_2\}$. Given sets E, $F\subset\overline{\mathbb{R}^n}$ and a domain $D\subset\mathbb{R}^n$ we denote by $\Gamma(E,F,D)$ a family of all paths $\gamma:[a,b]\rightarrow\overline{\mathbb{R}^n}$ such that $\gamma(a)\in E,\gamma(b)\in F$ and $\gamma(t)\in D$ for $t\in(a,b)$. If $f:D\rightarrow\mathbb{R}^n$, $y_0\in f(D)$ and $0<r_1<r_2<d_0=\sup\limits_{y\in f(D)}|y-y_0|$, then by $\Gamma_f(y_0,r_1,r_2)$ we denote the family of all paths γ in D such that $f(\gamma)\in\Gamma(S(y_0,r_1),S(y_0,r_2),A(y_0,r_1,r_2))$. Let $Q:\mathbb{R}^n\rightarrow[0,\infty]$ be a Lebesgue measurable function. We say that f *satisfies Poletsky inverse inequality* at the point $y_0\in f(D)$, if the relation

$$M(\Gamma_f(y_0,r_1,r_2))\leqslant\int\limits_{A(y_0,r_1,r_2)\cap f(D)}Q(y)\cdot\eta^n(|y-y_0|)\,dm(y)\qquad(1)$$

holds for any Lebesgue measurable function $\eta:(r_1,r_2)\rightarrow[0,\infty]$ such that

$$\int\limits_{r_1}^{r_2}\eta(r)\,dr\geqslant 1\,.\qquad(2)$$

Recall that a mapping $f:D\rightarrow\mathbb{R}^n$ is called *discrete* if the pre-image $\{f^{-1}(y)\}$ of each point $y\in\mathbb{R}^n$ consists of isolated points, and *is open* if the image of any open set $U\subset D$ is an open set in $\mathbb{R}^n$. Later, in the extended space $\overline{\mathbb{R}^n}=\mathbb{R}^n\cup\{\infty\}$ we use the spherical (chordal) metric h (see [3, Definition 12.1]). Further, the closure $\overline{A}$ and the boundary ∂A of the set $A\subset\overline{\mathbb{R}^n}$ we understand relative to the chordal metric h in $\overline{\mathbb{R}^n}$.

The boundary of D is called *weakly flat* at the point $x_0\in \partial D$, if for every $P>0$ and for any neighborhood U of the point x_0 there is a neighborhood $V\subset U$ of the same point such that $M(\Gamma(E, F, D))>P$ for any continua $E, F\subset D$ such that $E\cap \partial U\ne\varnothing\ne E\cap \partial V$ and $F\cap \partial U\ne\varnothing\ne F\cap \partial V$. The boundary of D is called weakly flat if the corresponding property holds at any point of the boundary D. Consider the following definition, see e.g. [1]. The boundary of a domain D in $\mathbb{R}^n$ is said to be *locally quasiconformal* if every $x_0\in\partial D$ has a neighborhood U that admits a quasiconformal mapping φ onto the unit ball $\mathbb{B}^n\subset\mathbb{R}^n$ such that $\varphi(\partial D\cap U)$ is the intersection of $\mathbb{B}^n$ and a coordinate hyperplane. The sequence of cuts σ_m, $m=1,2,\ldots,$ is called *regular,* if $\overline{\sigma_m}\cap\overline{\sigma_{m+1}}=\varnothing$ for $m\in\mathbb{N}$ and, in addition, $d(\sigma_m)\rightarrow 0$ as $m\rightarrow\infty$. If the end K contains at least one regular chain, then K will be called *regular.* We say that a bounded domain D in $\mathbb{R}^n$ is *regular*, if D can be quasiconformally mapped to a domain with a locally quasiconformal boundary whose closure is a compact in $\mathbb{R}^n$, and, besides that, every prime end in D is regular. Note that space $\overline{D}_P=D\cup E_D$ is metric, which can be demonstrated as follows. If $g:D_0\rightarrow D$ is a quasiconformal mapping of a domain D_0 with a locally quasiconformal boundary onto some domain D, then for $x, y\in \overline{D}_P$ we put:

$$\rho(x, y):=|g^{\,-1}(x)-g^{\,-1}(y)|\,, \tag{3}$$

where the element $g^{\,-1}(x)$, $x\in E_D$, is to be understood as some (single) boundary point of the domain D_0. The specified boundary point is unique, see e.g. [1, Theorem 4.1]. It is easy to verify that ρ in (3) is a metric on $\overline{D}_P$.

Let $E\subset\overline{D}$. We say that D is *finitely connected at the point* $z_0\in E$, if for each neighborhood $\widetilde{U}$ of z_0 there is a neighborhood $\widetilde{V}\subset\widetilde{U}$ of z_0 such that $(D\cap\widetilde{V})\setminus E$ consists of finite number of components. We say that D is *finitely connected on* E, if D is finitely connected at every point $z_0\in E$. The following theorem is true.

Theorem 1. *Let D and $D^{\,\prime}$ be domains in $\mathbb{R}^n$, $n\geqslant 2$, and let D be a domain with a weakly flat boundary. Suppose that f is open discrete mapping of D onto $D^{\,\prime}$ satisfying the relation (1) at each point $y_0\in\overline{D^{\,\prime}}$. In addition, assume that the following conditions are fulfilled:*

1) for each point $y_0\in\partial D^{\,\prime}$ there is $0<r_0:=\sup\limits_{y\in D^{\,\prime}}|y-y_0|$ such that for any $0<r_1<r_2<r_0:=\sup\limits_{y\in D^{\,\prime}}|y-y_0|$ there exists a set $E\subset[r_1, r_2]$ of positive linear Lebesgue measure such that Q is integrable on $S(y_0, r)$ for $r\in E$;

2) $D^{\,\prime}$ is a regular domain and, in addition, $D^{\,\prime}$ is finitely connected on $C(f, \partial D)\cap D^{\,\prime}$, i.e., for each point $z_0\in C(f, \partial D)\cap D^{\,\prime}$ and for any neighborhood U of this point there exists a neighborhood $V\subset U$ of this point such that the set $V\setminus C(f, \partial D)$ consists of a finite number of components;

3) the set $f^{\,-1}(C(f, \partial D)\cap D^{\,\prime})$ is nowhere dense in D;

4) the set $D^{\,\prime}$ is finitely connected in $E_{D^{\,\prime}}:=\overline{D^{\,\prime}}_P\setminus D^{\,\prime}$, i.e., for any $P_0\in E_{D^{\,\prime}}$ and any neighborhood U of P_0 in $\overline{D^{\,\prime}}_P$ there is a neighborhood $V\subset U$ such that $V\setminus C(f, \partial D)$ consists of finite number of components.

Then the mapping f has a continuous extension $\overline{f}:\overline{D}\rightarrow\overline{D^{\,\prime}}_P$ by the metric ρ defined in (3). Moreover, $\overline{f}(\overline{D})=\overline{D^{\,\prime}}_P$.

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Tree maps with acyclic Markov graphs

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Markov graphs provide an interesting tool in combinatorial dynamics, which helps to establish Sharkovsky-type results for continuous vertex maps on topological trees [1].

From purely discrete point of view, the construction of Markov graphs stems from a given vertex self-map on a combinatorial tree. Namely, let X be a tree and $\sigma : V(X) \to V(X)$ be a map. The corresponding Markov graph is a directed graph having the edge set $E(X)$ as its vertex set, with the arc set $\{(uv, xy) : x, y \in [\sigma(u), \sigma(v)]_X\}$ (here $[a, b]_G = \{x \in V(G) : d_G(a, x)+d_G(x, b) = d_G(a, b)\}$ denotes the metric interval between a, b in a connected graph G).

In other words, the vertices of $\Gamma(X, \sigma)$ are the edges of X with the existence of an arc $uv \to xy$ if only if uv "covers" xy under the map σ.

Here we are interested in maps on trees with acyclic Markov graphs. Note that tree maps with irreflexive Markov graphs are called anti-expansive. It can be proved that each anti-expansive map has a unique fixed point [2]. In case of acyclic Markov graphs, we can say much more.

Theorem 1. *Let X be a tree and $\sigma\colon V(X) \to V(X)$ be its vertex self-map. Then $\Gamma(X, \sigma)$ is acyclic if and only if there exists a "filtration" of subtrees $X = X_0 \supset X_1 \supset \cdots \supset X_m$ such that*

(1) $V(X_m) = \{u_0\}$ is a singleton with u_0 being a fixed point for f;
(2) $\sigma(V(X_k)) \subseteq V(X_{k+1})$ for $0 \le k \le m-1$.

A map $\sigma : V \to V$ is called nilpotent if there is $k \ge 1$ such that σ^k is constant. The next result completely describes the dynamical structure of maps on trees with acyclic Markov graphs.

Proposition 2. *A map $\sigma : V \to V$ is nilpotent if and only if there is a tree X on V such that $\Gamma(X, \sigma)$ is acyclic.*

In [3], the characterization of trees X which admit maps σ with $\Gamma(X, \sigma)$ being a path was obtained (these are the so-called balanced spiders).

A digraph is called an M-graph provided it is isomorphic to some Markov graph for a map on a tree. It can be proved that in-trees, out-trees, orientations of paths and stars are all M-graphs. However, not every polytree is an M-graph. To see this, consider the spider T obtained by gluing three copies of P_3 by their leaf vertices. Let D denotes the bipartite orientation of T in which the center of X becomes a source. Then it can be showed that D is not an M-graph.

Conjecture: Any polytree with out-degrees bounded by 2 is an M-graph.

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Integral problem for system of partial differential equations of higher order

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Let $H(\mathbb{R}_+ \times \mathbb{R}^n)$ be a class of entire functions on $\mathbb{R}$, K_L is a class of quasipolynomials of the form $\varphi(x) = \sum_{i=1}^n Q_r(x) \exp[\alpha_r x]$, where $\alpha_r \in L \subseteq \mathbb{C}$, $\alpha_k \neq \alpha_l$, for $k \neq l$, $Q_r(x)$ are given polynomials. In the strip $\Omega = \{(t,x) \in \mathbb{R}^{n+1} : t \in \{(0,T), x \in \mathbb{R}^n\}$, we consider of the system of equations

$$\frac{\partial^n U_i}{\partial t^n} + \sum_{j=1}^n a_{ij}\left(\frac{\partial}{\partial x}\right)\frac{\partial^{n-j}U_i}{\partial t^{n-j}} = 0, \tag{1}$$

$$\int\limits_0^T t^{n-j}U_i(t,x)dt = \varphi_{ik}(x), \quad k = \{1,...,n\}, \quad t \in [0,T], x \in \mathbb{R}^n. \tag{2}$$

Where $a_{ij}\left(\frac{\partial}{\partial x}\right)$, are differential expression with entire symbols $a_{ij}(\lambda) \neq 0$. Denote be P seet zeros of function $\eta(\lambda) = \int_0^T W^{n-1}(t,\lambda)dt$.

Theorem 1. ***Theorem.*** *Let $\varphi_{ik}(x) \in K_L$, $i = \{1,...,n\}$, $j = \{1,...,n\}$ then the class $K_{L\setminus P}$ exist and unique solution of the problem (1)-(2), can be represented in the form*

$$U_i(t,x) = \sum_{k=0}^{n-1}\sum_{p=1}^{n}\varphi_{kp}\left(\frac{\partial}{\partial x}\right)\left\{\frac{1}{\eta(\lambda)}T_{kjp}(t,\lambda)W(t,\lambda)\exp[\lambda x]\right\}\Bigg|_{\lambda=0},$$

Solution of the problem (1)-(2) according to the differential-symbol method [1,2],

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Homotopy properties of Morse functions on non-orientable surfaces

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Let M be a smooth compact surface and P is a real line $\mathbb{R}$ or a circle S^1. Denote by $\mathcal{F}(M,P)$ the space of smooth functions $f \in C^\infty(M,P)$ satisfying the following conditions:

1) the function f takes constant value at ∂M and has no critical point in ∂M;
2) for every critical point z of f there is a local presentation $f_z\colon \mathbb{R}^2 \to \mathbb{R}$ of f near z such that f_z is a homogeneous polynomial $\mathbb{R}^2 \to \mathbb{R}$ without multiple factors.

Let X be a closed subset of M. Denote by $\mathcal{D}(M, X)$ the group of C^∞-diffeomorphisms of M fixed on X, that acts on the space of smooth functions $C^\infty(M, P)$ by the rule: $(f, h) \longmapsto f \circ h$, where $h \in D(M, X)$, $f \in C^\infty(M, P)$.

Let $\mathcal{O}(f, X) = \{f \circ h \mid h \in D(M, X)$ be *orbit* of f with respect to the action above.

Precise algebraic structure of such orbits for oriented surfaces was described in [1]. In particular, the following theorem holds.

Theorem 1. [1] *Let M be a connected compact oriented surface except 2-sphere and 2-torus and let $f \in \mathcal{F}(M, P)$. Then $\pi_1\mathcal{O}(f, \partial M) \in \mathcal{G}$, where $\mathcal{G}$ is a minimal class of groups satisfying the following conditions:*

1) $1 \in \mathcal{G}$;
2) if $A, B \in \mathcal{G}$, then $A \times B \in \mathcal{G}$;
3 if $A \in \mathcal{G}$ and $n \geq 1$, then $A \wr_n \mathbb{Z} \in \mathcal{G}$.

Definition 2. Let G, H be groups, $m \in \mathbb{Z}$ and $\gamma\colon H \to H$ be automorphism of order 2. Define the automorphism $\phi\colon G^{2m} \times H^m \to G^{2m} \times H^m$ by the formula

$$\phi(g_0, \dots, g_{2m-1}, h_0, \dots, h_{m-1}) = (g_{2m-1}, g_0, \dots, g_{2m-2}, h_1, h_2, \dots, h_{m-1}, \gamma(h_0)).$$

This automorphism ϕ generates homomorphism $\phi'\colon \mathbb{Z} \to G^{2m} \times H^m$. The corresponding semidirect product $G^{2m} \times H^m \rtimes_{\phi'} \mathbb{Z}$ will be denoted $(G, H) \wr_{\gamma,m} \mathbb{Z}$.

Theorem 3. [2] *Let M be a Möbius band. Then for every $f \in \mathcal{F}(M, P)$ either*

(1) exist groups $A, G, H \in \mathcal{G}$, an authmorphism $\gamma\colon H \to H$ of order 2 and $m \geq 1$, such that

$$\pi_1\mathcal{O}(f, \partial M) \cong A \times (G, H) \wr_{\gamma,m} \mathbb{Z},$$

(2) or there exist groups $A, G \in \mathcal{G}$ and odd number $m \geq 1$ such that

$$\pi_1\mathcal{O}(f, \partial M) \cong A \times G \wr_b \mathbb{Z}.$$

Conversely, for every such tuple (A, G, m) or (A, G, H, γ, m) there exists $f \in \mathcal{F}(M, P)$ such that we have the corresponding isomorphism.

It was shown in [1] that if M has negative Euler characterictic, then fundamental groups of orbits of functions in $\mathcal{F}(M, P)$ are direct products of such groups for functions only on cylinders, disks and Möbius bands.

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The solution to a problem in linear chain recurrence

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We exhibit the existence of continuous (and even invertible) linear operators acting on Banach (and even Hilbert) spaces whose restriction to their respective closed linear subspaces of chain recurrent vectors are not chain recurrent operators. This example completely solves in the negative a problem posed in [1] by N. C. Bernardes Jr. and A. Peris on chain recurrence in Linear Dynamics. The results exhibited along this talk can be found in [2], which is a joint work with Dimitris Papathanasiou.

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On some dynamical systems given in terms of a chain A_2-representation

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Let $0 < \alpha_1 < \alpha_2$, $\alpha_1\alpha_2 = \frac{1}{2}$ and

$$[a_0; a_1, a_2, \dots, a_k, \dots] = a_0 + \cfrac{1}{a_1 + \dots + \cfrac{1}{a_{k-1} + \cfrac{1}{a_k + \dots}}}$$

— continued fraction, where $a_0 \in Z_+$, $a_j > 0$ for all $j \in N$. It is well known from [1] that for any $t \in [\alpha_1; \alpha_2]$, there exists a sequence (b_n) with $b_n \in \{\alpha_1; \alpha_2\}$ for all $n \in \mathbb{N}$, such that

$$t = [0; b_1, b_2, \dots; b_n, \dots].$$

This expression is called the A_2-representation with the alphabet $\{\alpha_1; \alpha_2\}$. A countable subset of $[\alpha_1; \alpha_2]$ has two distinct A_2-representations of the form

$$[0; b_1, \dots; b_n, \alpha_1, (\alpha_1; \alpha_2)] = [0; b_1, \dots, b_n, \alpha_2, (\alpha_2; \alpha_1)],$$

where parentheses denote the period of a given continued fraction. Numbers possessing the above property are called A_2-binary. In contrast, numbers in the interval $[\alpha_1; \alpha_2]$ that are not A_2-binary and have a unique A_2-representation are termed A_2-unary. A detailed analysis of the topological and metric properties of the A_2-representation is provided in [1].

Consider left shift operator

$$T([0; a_1, a_2, ..., a_n, ...]) = [0; a_2, a_3, ..., a_{n+1}, ...].$$

From now on, we agree not to use A_2-representations with period $(\alpha_1; \alpha_2)$ for A_2-binary numbers.

Let (ξ_n) be a sequence of independent discretely distributed random variables that take values α_1 or α_2 with probabilities $\rho \in (0;1)$ and $1-\rho$, respectively. Let $\eta(\cdot)$ be the Lebesgue-Stieltjes measure corresponding to the distribution

$$\xi = [0; \xi_1, \xi_2, \dots, \xi_n, \dots].$$

The Lebesgue structure of the distribution of the random variable ξ was studied in [2]. The new result is the following.

Theorem 1. *The following statements are true:*

1. The dynamical system $([\alpha_1;\alpha_2]; B(\mathbb{R}) \cap [\alpha_1;\alpha_2]; T; \eta(\cdot))$ *is ergodic and the transformation* T *is strongly mixing:*

$$\lim_{n\to+\infty} \eta(T^{-n}(A) \cap B) = \eta(A)\eta(B) \quad \forall A, B \in B(\mathbb{R}) \cap [\alpha_1;\alpha_2];$$

2. The entropy of $([\alpha_1;\alpha_2]; B(\mathbb{R}) \cap [\alpha_1;\alpha_2]; T; \eta(\cdot))$ *is equal to*

$$h(T) = 2\ln\left(\frac{\rho\alpha_1 + (1-\rho)\alpha_2 + \sqrt{(\rho\alpha_1 + (1-\rho)\alpha_2)^2 + 4}}{2}\right).$$

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Classification of smooth structures on non-Hausdorff one-dimensional manifolds

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Previously [1] the authors gave a classification of C^r-differentiable structures, $r > 0$, on the non-Hausdorff line L with two origins. The aim of the present talk is to give a classification of differentiable structures the non-Hausdorff one-dimensional manifold Y called *non-Hausdorff letter* Y.

It turns out that in contrast with the real line those manifolds have infinitely many pair-wise non-diffeomorphic structures.

Moreover, the arguments of both proofs are similar and can be given only in terms of certain commutative diagrams. In particular, this allows to extends arguments to the following general problem. Given a pair of integers $0 < s < r \leq \infty$, it is possible to classify C^r structures of L (or Y) up to a C^s-diffeomorphism.

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Equivalence groupoids and group classification of (1+3)-dimensional nonlinear Schrödinger equations

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The study of Lie symmetries of nonlinear Schrödinger equations was started in the late 1970es and was then continued by many scientists, see [1, 2, 3] and references therein. An important class $\mathcal{V}$ of $(1+n)$-dimensional nonlinear Schrödinger equations with modular nonlinearities and complex-valued potentials was comprehensively considered within the framework of Lie symmetries in the literature, but the main results in this direction were obtained initially in [3] and then in [1]. The above equations are the form

$$i\psi_t + \psi_{aa} + f(\rho)\psi + V(t,x)\psi = 0, \tag{1}$$

where t and $x = (x_1, \dots, x_n)$ are the real independent variables, $n \in \mathbb{N}$, ψ is the unknown complex-valued function of (t,x), V is an arbitrary smooth complex-valued potential depending on (t,x), and f is an arbitrary complex-valued nonlinearity depending only on $\rho := |\psi|$, $f_\rho \neq 0$. Subscripts of functions denote differentiation with respect to the corresponding variables. The index a runs from 1 to n, and summation over repeated indices is assumed. Particularly known equations from the class $\mathcal{V}$ are cubic Schrödinger equations with potentials, where $f(\rho) = \rho^2$. The complete group classification of this class was carried out in [3] for $n = 1$ and in [1] for $n = 2$. Moreover, the last reference also contains preliminary results on group analysis of the class $\mathcal{V}$ for the case of arbitrary n. At the same time, even in the most physically relevant case $n = 3$, the problem of complete group classification of the class $\mathcal{V}$ is still open. The class $\mathcal{V}$ is not normalized, but it can be partitioned into three disjoint normalized subclasses, which are not related to each other by point transformations. These are the subclasses with logarithmic, power and general nonlinearities. We started extending the results of [1, 3] to the case $n = 3$ and were able to carry out the major part of the group classification for $n = 3$ and general modular nonlinearities.

Denote by $\mathcal{V}^f$ the subclass of the class $\mathcal{V}$ with $n = 3$ and a fixed general value of the nonlinearity f, i.e., $\rho f_{\rho\rho}/f_\rho$ is not a real constant, by ψ^* the complex conjugate of ψ,

$$D(1) := \partial_t, \quad J_1 := x_2\partial_3 - x_3\partial_2, \quad J_2 := x_3\partial_1 - x_1\partial_3, \quad J_3 := x_1\partial_2 - x_2\partial_1,$$
$$P(\boldsymbol{\chi}) := \chi^a\partial_a + \frac{1}{2}\chi^a_t x_a M, \quad M := i\psi\partial_\psi - i\psi^*\partial_{\psi^*},$$

where the parameters χ^a and σ are real-valued smooth functions of t.

Lemma 1. *The class $\mathcal{V}^f$ is normalized. The maximal Lie invariance algebra $\mathfrak{g}_V$ of an equation $\mathcal{L}_V$ from this class consists of the vector fields of the form $cD(1) - \kappa_a J_a + P(\boldsymbol{\chi}) + \sigma M$, where c and κ_a*

are arbitrary real constants and the parameter functions χ^a *and* σ *are arbitrary real-valued smooth functions of* t *that satisfy the condition*

$$cV_t + (\kappa_2 x_3 - \kappa_3 x_2 + \chi^1)V_1 + (\kappa_3 x_1 - \kappa_1 x_3 + \chi^2)V_2 + (\kappa_1 x_2 - \kappa_2 x_1 + \chi^3)V_3 = \frac{1}{2}\chi^a_{tt}x_a + \sigma_t. \qquad (2)$$

The kernel Lie invariance algebra of equations from the class $\mathcal{V}^f$ *is* $\mathfrak{g}^{\cap}_{\mathcal{V}^f} := \langle M\rangle$.

Any vector field of the general form presented in Lemma 1, where at least one of the parameters c, κ_a and χ^a takes a nonzero value, belongs to $\mathfrak{g}_V$ for a potential V satisfying the classifying condition (2) for this vector field. This is why we have $\mathfrak{g}_{\langle\rangle} := \sum_V \mathfrak{g}_V = \langle\, D(1),\, J_a,\, P(\boldsymbol{\chi}),\, \sigma M\,\rangle$, where the parameter functions χ^a and σ run through the set of real-valued smooth functions of t,

A subalgebra $\mathfrak{s}$ of $\mathfrak{g}_{\langle\rangle}$ is called *appropriate* if $\mathfrak{s} = \mathfrak{g}_V$ for some V. For each of such subalgebras, we define five nonnegative integers that depend on V, are invariant under equivalence transformations of the class $\mathcal{V}^f$ and label the cases of Lie-symmetry extensions within this class,

$$\begin{aligned}
&r_1 := \operatorname{rank}\{\chi \mid \exists\,\sigma\colon P(\boldsymbol{\chi}) + \sigma M \in \mathfrak{s}\}, \quad k_0 := \dim \mathfrak{s}\cap\langle\sigma M\rangle = \dim \mathfrak{g}^{\cap} = 1,\\
&k_1 := \dim \mathfrak{s}\cap\langle P(\boldsymbol{\chi}),\, \sigma M\rangle - k_0, \quad k_2 := \dim \mathfrak{s}\cap\langle J_1,\, J_2,\, J_3,\, P(\boldsymbol{\chi}),\, \sigma M\rangle - k_1 - k_0,\\
&k_3 := \dim \mathfrak{s} - k_2 - k_1 - k_0.
\end{aligned}$$

One has $r_1 \in \{0,1,2,3\}$, $k_2 \in \{0,1,3\}$, $r_1 \leqslant k_1$, $k_1 \in \{0,\dots,6\}$ and $k_3 \in \{0,1\}$ [1, Section 6].

The following lemmas are useful in the course of the group classification of the class $\mathcal{V}^f$ with $n = 3$.

Lemma 2. *(i) If* $k_2 = 3$ *and* $Q^0 = P(\boldsymbol{\chi}^0) + \sigma^0 M + \zeta^0 I \in \mathfrak{s}$*, then* $P(\chi^{0a}\boldsymbol{\delta}_b) + \breve\sigma^{ab}M \in \mathfrak{s}$ *for any* $a, b \in \{1,2,3\}$ *and some functions* $\breve\sigma^{ab}$ *of* t.

(ii) If $k_2 = 3$*, then* $\mathfrak{s} \supseteq \langle J_1,\, J_2,\, J_3\rangle$ *modulo the point equivalence in the class* $\mathcal{V}^f$*, and* $r_1 \in \{0,3\}$.

Lemma 3. *If* $\boldsymbol{\chi}$ *and* $\breve{\boldsymbol{\chi}}$ *are linearly independent and* $\boldsymbol{\chi}\cdot\breve{\boldsymbol{\chi}}_t - \boldsymbol{\chi}_t\cdot\breve{\boldsymbol{\chi}} = 0$*, then* $\operatorname{rank}(\boldsymbol{\chi}, \breve{\boldsymbol{\chi}}) = 2$.

Lemma 4. *If* $\operatorname{rank}(\boldsymbol{\chi}^1, \boldsymbol{\chi}^2) = \operatorname{rank}(\boldsymbol{\chi}^1, \boldsymbol{\chi}^2, \boldsymbol{\chi}) = 2$ *and* $\boldsymbol{\chi}\cdot\boldsymbol{\chi}^l - \boldsymbol{\chi}_t\cdot\boldsymbol{\chi}^l = 0$*,* $l = 1,2$*, then* $\boldsymbol{\chi} \in \langle\boldsymbol{\chi}^1, \boldsymbol{\chi}^2\rangle$.

Lemma 5. *Let* $r_1 = 2$*, i.e., the algebra* $\mathfrak{g}_V$ *contains at least two vector fields of the form* $Q^s = P(\boldsymbol{\chi}^s) + \sigma^s M + \zeta^s I$*,* $s = 1,2$*, where* $\operatorname{rank}(\boldsymbol{\chi}^1, \boldsymbol{\chi}^2) = 2$ *for any* t *in the related interval. Denote* $\boldsymbol{\chi}^0 := \boldsymbol{\chi}^1 \times \boldsymbol{\chi}^2 \neq \mathbf{0}$*. Then* $\boldsymbol{\chi}^1\cdot\boldsymbol{\chi}^2_t - \boldsymbol{\chi}^1_t\cdot\boldsymbol{\chi}^2 = \mathrm{const}$ *and the following holds:*

(i) $k_1 \in \{2,3,4\}$.

(ii) $k_1 = 2$ *if and only if* $\big(\boldsymbol{\chi}^0\times\boldsymbol{\chi}^0_t,\, \boldsymbol{\chi}^0_{tt} + \boldsymbol{\chi}^1_t\times\boldsymbol{\chi}^2_t\big) \neq 0$.

(iii) $k_1 = 3$ *if and only if* $\big(\boldsymbol{\chi}^0\times\boldsymbol{\chi}^0_t,\, \boldsymbol{\chi}^0_{tt} + \boldsymbol{\chi}^1_t\times\boldsymbol{\chi}^2_t\big) = 0$ *but* $\boldsymbol{\chi}^0\times\boldsymbol{\chi}^0_t \neq \mathbf{0}$.

(iv) $k_1 = 4$ *if and only if* $\boldsymbol{\chi}^0\times\boldsymbol{\chi}^0_t = \mathbf{0}$.

We have classified the equations from the class $\mathcal{V}^f$ with $r_1 \in \{0,1,3\}$ and are almost complete classification for the case $r_1 = 2$. In particular, if $r_1 = 3$, $k_2 = 0$, $k_3 = 1$, then up to the point equivalence within the class $\mathcal{V}^f$, the algebra $\mathfrak{g}_V$ necessary contains the vector field $D(1) + \kappa J_3$ with $\kappa = \mathrm{const}$. For nonzero κ, the corresponding case of Lie symmetry extension is the following:

$$\begin{aligned}
&V = \tfrac{1}{4}(\alpha_1\omega_1^2 + \alpha_2\omega_2^2 + \alpha_3 x_3^2) + \tfrac{1}{2}(\beta_1\omega_1 + \beta_2\omega_2)x_3 + i\nu\colon\\
&\mathfrak{g}_V = \langle M,\, P(\theta^{p1}\cos\kappa t - \theta^{p2}\sin\kappa t,\, \theta^{p1}\sin\kappa t + \theta^{p2}\cos\kappa t,\, \theta^{p3}),\, p = 1,\dots,6,\, D(1) + \kappa J_3\rangle,
\end{aligned}$$

where $\omega_1 := x_1\cos\kappa t + x_2\sin\kappa t$, $\omega_2 := -x_1\sin\kappa t + x_2\cos\kappa t$, $\omega_3 := x_3$; α_1, α_2, β_1, β_2 and κ are real constants with $\alpha_2 \neq \alpha_1 \neq 0$, and $\kappa \neq 0$; $(\theta^{p1}(t), \theta^{p2}(t), \theta^{p3}(t))$ are linearly independent solutions of the system,

$$\theta^1_{tt} - 2\kappa\theta^2_t = (\kappa^2 + \alpha_1)\theta^1 + \beta_1\theta^3, \quad \theta^2_{tt} + 2\kappa\theta^1_t = (\kappa^2 + \alpha_2)\theta^2 + \beta_2\theta^3, \quad \theta^3_{tt} = \alpha_3\theta^3.$$

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Symplectic representation of surface mapping classes of algebraically finite type

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In [4] Nielsen investigated properties of surface mapping classes of algebraically finite type, defined to be represented by homeomorphisms that are either periodic or reducible and periodic outside an invariant system of circles. In other words, they have no pseudo-Anosov pieces in their Nielsen–Thurston decomposition. The name ”algebraically finite type” was motivated by Nielsen's conjecture that such classes can be defined purely algebraically as the ones that induce a map on the first homology group whose spectrum consists only of roots of unity (the latter classes are called quasi-unipotent). These two definitions do not coincide because of Thurston's construction of pseudo-Anosov map inducing the identity transformation. However, the following question is still open: which symplectic transformations can be obtained from mapping classes of algebraically finite type? In particular, what is the maximum finite order of such symplectic matrices? We will discuss this problem, important also from the point of view of dynamics. Da Rocha [1] showed that the classes containing Morse–Smale diffeomorphism and classes of algebraically finite type are the same. Some constructions in terms of Lefschetz numbers we provided in [2, 3].

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The Arithmetic and Geometric Properties of Rational Surfaces

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Rational surfaces are an important research subject in surface theory. The surfaces with minimal degree and Del Pezzo surfaces are classical examples of rational surfaces. In this talk, we will focus on recent advances in the study of rational surfaces. In particular, we will investigate the structure of rational surfaces from both arithmetic and geometric perspectives.

Theory of Galois covers is the geometric counterpart of classical Galois theory. In [1], we proved that the Galois covers of surfaces with minimal degree are simply connected surfaces. Furthermore, we also considered extending these results to Zappatic surfaces. We proved that:

Theorem 1. *If a smooth projective algebraic surface deforms to a Zappatic surface of type E_n, $n \geq 4$, then its Galois cover is simply connected and of general type.*

Furthermore, we have the following result:

Theorem 2. *For a complex algebraic surface $X \subset \mathbb{P}^2$, if there exists a degeneration X_0 such that X_0 has only singularities of type R_k, and X_0 has R_k loops, then the Galois cover of the surface X is not simply connected.*

The topological invariants of a surface are important invariants in surface theory. In this talk, we will also introduce the invariants of Galois covers, with a particular focus on Zappatic surfaces that have only singularities of type R_k:

Theorem 3. *The signature $\tau(X_{Gal}) = \frac{1}{3}n!\ (-I(T))$ where n is the degree of X, and $I(T)$ is the number of vertices of degree 2 in the dual graph T of X_0.*

We also extend this result to surface fibrations over rational surfaces, thereby obtaining some geometric properties of rational surfaces:

First, for elliptic fibrations, we have ([2]):

Theorem 4. *Given an rational elliptic surface S over $\mathbb{P}^1$, with the generic fibre F , we give the number of integral sections.*

Secondly, for fibrations with two singular fibers, we have ([3]).

Theorem 5. *Let $f : S \longrightarrow \mathbb{P}^1$ be a relatively minimal fibration of genus $g \geq 2$ with two singular fibers, F_1 and F_2.*

- *If $0 \neq \tau(S)$, then $\tau(S) \leq -4$.*
- *If $\tau(S) < -4$, then $\tau(S) \leq -6$.*

Finally, we will present some applications of the theorems discussed and propose several open problems.

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Surfaces with flat normal connection in 4-dimensional space forms

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Let N be a 4-dimensional Riemannian space form with constant sectional curvature L_0. Let h be the metric of N and ∇ the Levi-Civita connection of h. Let M be a Riemann surface and $F : M \longrightarrow N$ a conformal immersion. Let (u, v) be local isothermal coordinates of M. Then the induced metric of M by F is represented as $g = e^{2\lambda}(du^2 + dv^2)$ for a function λ. We set $T_1 := dF(\partial/\partial u)$, $T_2 := dF(\partial/\partial v)$. Let N_1, N_2 be normal vector fields of F satisfying $h(N_1, N_1) = h(N_2, N_2) = e^{2\lambda}$, $h(N_1, N_2) = 0$.

Suppose that N is oriented and that (T_1, T_2, N_1, N_2) gives the orientation. We set

$$e_1 := \frac{1}{e^\lambda}T_1, \quad e_2 := \frac{1}{e^\lambda}T_2, \quad e_3 := \frac{1}{e^\lambda}N_1, \quad e_4 := \frac{1}{e^\lambda}N_2$$

and

$$\Omega_{\pm,1} := \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \pm e_3 \wedge e_4), \quad \Omega_{\pm,2} := \frac{1}{\sqrt{2}}(e_1 \wedge e_3 \pm e_4 \wedge e_2),$$
$$\Omega_{\pm,3} := \frac{1}{\sqrt{2}}(e_1 \wedge e_4 \pm e_2 \wedge e_3).$$

The two-fold exterior power $\bigwedge^2 F^*TN$ of the pull-back bundle F^*TN on M by F is of rank 6 and decomposed into two subbundles $\bigwedge^2_\pm F^*TN$ of rank 3, and $\Omega_{\pm,1}$, $\Omega_{\pm,2}$, $\Omega_{\pm,3}$ form local orthonormal frame fields of $\bigwedge^2_\pm F^*TN$ respectively.

Let $\hat{\nabla}$ be the connection of $\bigwedge^2 F^*TN$ induced by ∇. Then $\hat{\nabla}$ gives connections of $\bigwedge^2_\pm F^*TN$ and we obtain

$$\begin{aligned}\hat{\nabla}_{T_1}(\Omega_{\pm,1}\ \Omega_{\pm,2}\ \Omega_{\pm,3}) &= (\Omega_{\pm,1}\ \Omega_{\pm,2}\ \Omega_{\pm,3})\begin{bmatrix} 0 & -W_\pm & -Y_\mp \\ W_\pm & 0 & \pm\psi_\pm \\ Y_\mp & \mp\psi_\pm & 0 \end{bmatrix}, \\ \hat{\nabla}_{T_2}(\Omega_{\pm,1}\ \Omega_{\pm,2}\ \Omega_{\pm,3}) &= (\Omega_{\pm,1}\ \Omega_{\pm,2}\ \Omega_{\pm,3})\begin{bmatrix} 0 & \mp Z_\pm & \pm X_\mp \\ \pm Z_\pm & 0 & \mp\phi_\mp \\ \mp X_\mp & \pm\phi_\mp & 0 \end{bmatrix}\end{aligned} \tag{1}$$

([4]), where

(i) $W_\pm$, $X_\pm$, $Y_\pm$, $Z_\pm$ are functions related to the second fundamental form σ of F satisfying $W_+ + W_- = X_+ + X_-$, $Y_+ + Y_- = Z_+ + Z_-$,

(ii) $\phi_\pm := \lambda_u \mp \mu_2$, $\psi_\pm := \lambda_v \mp \mu_1$, and μ_1, μ_2 are functions related to the normal connection $\nabla^\perp$ of the immersion F (in particular, if $\nabla^\perp$ is flat, then there exists a function γ satisfying $\gamma_u = \mu_1$, $\gamma_v = \mu_2$).

Let $\hat{R}$ be the curvature tensor of $\hat{\nabla}$. Then computing $\hat{R}(T_1, T_2)(\Omega_{\pm,1}\ \Omega_{\pm,2}\ \Omega_{\pm,3})$ by (1) and noticing that N is a space form of constant sectional curvature L_0, we obtain

$$W_{\mp}X_{\pm} + Y_{\pm}Z_{\mp} = L_0 e^{2\lambda} + (\phi_{\pm})_u + (\psi_{\mp})_v \tag{2}$$

and

$$\begin{aligned} (Y_{\pm})_v \mp (X_{\pm})_u &= \pm W_{\mp}\phi_{\pm} - Z_{\mp}\psi_{\mp}, \\ (W_{\mp})_v \pm (Z_{\mp})_u &= \mp Y_{\pm}\phi_{\pm} - X_{\pm}\psi_{\mp}. \end{aligned} \tag{3}$$

As in [5], (2) is equivalent to the system of the equations of Gauss and Ricci, and (3) is equivalent to the system of the equations of Codazzi.

In [5], immersions with flat normal connection are studied. Let $R^{\perp}$ be the curvature of the normal connection $\nabla^{\perp}$. By definition, $R^{\perp} = 0$ just means that $\nabla^{\perp}$ is flat. If F has a parallel normal vector field, then the second fundamental form σ satisfies the linearly dependent condition and then $\nabla^{\perp}$ is flat (see [5]). Suppose that the curvature K of g is nowhere equal to L_0. Then F has a parallel normal vector field if and only if σ satisfies the linearly dependent condition ([5]). On the other hand, if we suppose $K = L_0$, then the linearly dependent condition of σ does not necessarily mean the existence of parallel normal vector fields ([5]).

By (2), F satisfies $W_{\mp}X_{\pm} + Y_{\pm}Z_{\mp} = 0$ if and only if $R^{\perp} = 0$ and $K = L_0$ hold. Suppose that there exist functions $k_{\pm}$ satisfying

$$(W_{\mp}, Z_{\mp}) = k_{\pm}(-Y_{\pm}, X_{\pm}). \tag{4}$$

Then $W_{\mp}X_{\pm} + Y_{\pm}Z_{\mp} = 0$ hold. Applying (4) to (3), we see that there exist functions $f_{\pm}$ satisfying

$$X_{\pm} = \pm\frac{(f_{\pm})_v}{\sqrt{1+k_{\pm}^2}}, \quad Y_{\pm} = \frac{(f_{\pm})_u}{\sqrt{1+k_{\pm}^2}}, \tag{5}$$

and by the equation of Ricci, we obtain $(f_+)_u^2 + (f_+)_v^2 = (f_-)_u^2 + (f_-)_v^2$. Therefore, if we suppose $(f_+)_u^2 + (f_+)_v^2 \neq 0$, then there exists a function ψ satisfying

$$\begin{bmatrix} (f_-)_u \\ (f_-)_v \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} (f_+)_v \\ (f_+)_u \end{bmatrix}. \tag{6}$$

Suppose that $k_{\pm}$ is nowhere zero and that $X_{\pm}$, $Y_{\pm}$ satisfy $X_+^2Y_-^2 - X_-^2Y_+^2 \neq 0$. Then the second fundamental form σ does not satisfy the linearly dependent condition ([5]), and using (3), (4) and (5), we can obtain an over-determined system for the function γ related to $\nabla^{\perp}$ ([5]). In addition, if we suppose $L_0 = 0$, then the compatibility condition of this over-determined system can be represented as an over-determined system of polynomial type with degree two for the function ψ in (6) ([5]). See [1] for over-determined systems of polynomial type.

In the above discussions, we supposed that N is a Riemannian space form. Suppose that N is a 4-dimensional neutral space form with constant sectional curvature L_0. Then for a Riemann or Lorentz surface M and a space-like or time-like, and conformal immersion $F : M \longrightarrow N$, we can have similar discussions and obtain analogous results ([4], [5]). See [2], [3] for time-like surfaces in N with zero mean curvature vector and $K \equiv L_0$ (such surfaces have flat normal connection). In the case where N is a 4-dimensional Lorentzian space form with constant sectional curvature L_0, noticing that the complex bundle $\bigwedge^2 F^*TN \otimes \boldsymbol{C}$ is decomposed into two subbundles of complex rank 3, we can have similar discussions and obtain analogous results ([4], [5]).

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On realizations of Lie algebras

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In this talk we survey and compare several methods of representation of Lie algebras by vector fields, namely we consider the direct method [2], the Shirokov's method [3], the Blattner's method [1] and standard weight representations [4]. We consider Lie algebras defined by their structure constants tensor in some fixed basis and study the problem of construction of all their realizations.

Definition 1. *A realization* of a Lie algebra $\mathfrak{g}$ in vector fields on a domain $M \subset \mathbb{C}^m$ (or $M \subset \mathbb{R}^m$) is a homomorphism $R\colon \mathfrak{g} \to \mathrm{Vect(M)}$.

The interest to this subject is motivated by a number of applications, in particular in classification and integration of differential equations, see [2] for more applications.

We present several illustrative examples, contrary instances and some new results concerning realizations of special linear Lie algebra.

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Chaotic properties of weighted shifts

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Let $T : X \mapsto X$ be a bounded linear operator acting on topological space X

Definition 1. The operator T is Li-Yorke chaotic if there is uncountable set $U \subset X$, called scrambled set, such that for each $x, y \in U$, $x \neq y$, $\lim\limits_{n\mapsto\infty} \|T^n(x) - T^n(y)\| = 0$

Let $(H_n)_{n=0}^{\infty}$ be a sequence of Hilbert spaces. Each space H_n is supposed to be nontrivial and possibly nonseparable.

Assume that for all n and m, the spaces H_n and H_m are isomorphic. We define $\ell_2(H_n) = \ell_2((H_n)_{n=0}^{\infty})$ as the Hilbert space consisting of elements $x = (x_0, x_1, \ldots, x_n, \ldots)$, $x_k \in H_k$ endowed with norm $\|x\| = (\sum_{i=0}^{\infty} \|x_i\|^2)^{\frac{1}{2}}$.

Let (ω_n) be a sequence of weights and let us fix a sequence of isomorphisms $J_m : H_m \to H_{m-1}$, $\|J_m\| = 1$, $m \in \mathbb{N}$

$$0 \longleftarrow H_0 \xleftarrow{J_1} H_1 \xleftarrow{J_2} \cdots \xleftarrow{J_n} H_n \cdots .$$

An operator

$$T\colon \ell_2(H_n) \to \ell_2(H_n)$$

will be called a *backward weighted shift (with respect to the family* (J_m)*) with weight sequence* (ω_n) if it is of the form

$$T(x) = (\omega_1 J_1(x_1), \omega_2 J_2(x_2), \ldots, \omega_m J_m(x_m), \ldots).$$

Theorem 2. *Let* $(H_n)_{n=0}^{\infty}$ *be a sequence of Hilbert spaces. A backward weighted shift* $T\colon \ell_2(H_n) \to \ell_2(H_n)$, $T(x) = (\omega_1 J_1(x_1), \omega_2 J_2(x_2), \ldots, \omega_m J_m(x_m), \ldots)$ *is Li-Yorke chaotic*

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Wilson loops as a device for studying phase transitions and conductivity effects

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Wilson loops have become a fundamental tool in high-energy physics. They have allowed the formulation and verification of key properties of quantum chromodynamics and laid the foundation for the numerical analysis of strong interactions. Their applications extend beyond QCD to broad areas of theoretical physics, including string theory and quantum gravity [1].

The importance of Wilson loops can be summarized in the following list:

• Proof of confinement – Wilson loops allowed us to quantitatively explain why quarks are not observed in a free state;

• Development of lattice QCD – this approach made it possible to simulate strong interactions on computer;

• Application in string theory – Wilson loops are associated with the so-called "string break" effects and the formation of "color tubes" between quarks;

• Generalizations to other theories used in various gauge theories, including gravity and condensed matter.

There are two most important aspects of the application of Wilson loops that need to be addressed:
1. The topological aspect:
In systems with topological order, the conductivity can be related to topological invariants that are expressed in terms of Wilson loops. For example, in the quantum Hall effect, the conductivity is quantized, which can be described using nontrivial topological configurations of Wilson loops.
2. Lattice QCD:
In lattice QCD calculations, Wilson loops are used to study the potentials between quarks and the properties of quark-gluon plasma. The plasma conductivity depends on the mobility of quarks and gluons, whose interactions are encoded in the Wilson loops.

For a quantized magnetic field, $\Phi = 2\pi n/e$ the Wilson loop takes the value

$$W(c) = e^{i2\pi n}.$$

For a magnetic monopole with a magnetic charge g at the center of the field of radius R the Wilson loop takes the value

$$W(C) = e^{2\pi g/R}.$$

Quantization of magnetic flux leads to discrete values of the Wilson loop, and quantization o magnetic charge indicates the topological nature of the monopole.

A powerful tool that combines physics and mathematics, Chern-Simons theory allows us to study the topological properties of systems and their stability [2]. Chern-Simons theory describes topological invariants that do not depend on the metric of the space, but only on its topological structure. The basis of the theory is the Chern-Simons action, which is defined in three-dimensional space and depends on the gauge field. It has the form:

$$S = \frac{k}{4\pi} \int tr \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

average value of Wilson loop

$$< W_R(C) >= \frac{1}{Z} \int DAW_R(C) e^{iS(A)},$$

where $Z = \int DAe^{iS(A)}$ is a statistical sum.

Chern-Simons theory plays a key role in describing topological phases of matter, such as the fractional quantum Hall effect (FQHE) [3]. The Laughlin states are the most well-known examples of fractional quantum Hall states, occurring at filling factors $\nu = 1/m$, where m is an odd integer (e.g., $\nu = 1/3, 1/5, \ldots$). In deriving the topological response of the Laughlin state, particularly the quantized Hall conductivity let's obtain the effective action for the external electromagnetic field A_μ connected with the Lagrangian

$$L = \frac{m}{4\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \frac{e}{2\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu a_\rho.$$

To derive the Hall conductivity from the effective action, we start with the effective Chern-Simons action for the external electromagnetic field A_μ:

$$L_{eff} = \frac{e^2}{4\pi m} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho.$$

This action describes the topological response of the system to the external field A_μ. The coefficient $\frac{e^2}{4\pi m}$ encodes the filling factor $\nu = 1/m$ of the Laughlin state. Using Hall conductivity expression $\sigma_{xy} = \frac{j^x}{E_y}$ and substituting $j^\mu = \frac{\delta L_{eff}}{\delta A_\mu}$, we get

$$\sigma = \frac{e^2}{4\pi m} = \frac{e^2}{h}\nu.$$

This derivation shows how the topological Chern-Simons term in the effective action leads to the quantized Hall conductivity, a hallmark of the Laughlin state.

To find the phase factor for exchanging two quasiparticles in the Laughlin state, we need to analyze the braiding statistics of quasiparticles using the framework of Chern-Simons theory and Wilson loops. Consider two quasiparticles at positions r_1 and r_2. When the quasiparticles are exchanged, their trajectories form a braid in spacetime. The phase factor associated with this exchange can be computed using the linking number of the Wilson loops $< W(C_1)W(C_2) >= e^{i\frac{2\pi}{m}} Link(C_1, C_2)$. For a simple exchange of two quasiparticles, the linking number is $Link(C_1, C_2) = 1$ and

$$< W(C_1)W(C_2) >= e^{i\frac{2\pi}{m}}.$$

Exchanging two quasiparticles corresponds to half of a full braid and we have

$$e^{i\frac{\pi}{m}}.$$

The phase factor signals about:
1. The Chern-Simons term in the effective action enforces the fractional braiding phase.
2. The exchange of two quasiparticles corresponds to a π rotation in spacetime, leading to the phase factor $e^{i\pi/m}$.
3. The linking number of Wilson loops describing quasiparticle trajectories.

From the results of comparing the expressions for the Wilson loop and conductivity, we see proportionality of the average value of Wilson loop to the filling factor $\nu = 1/m$. The movement of one quasiparticle around another produces a topological phase shift associated with the charge and statistics of the particles.

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Translation length formula for two-generated groups acting on trees

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The notion of a *tree* is considered at various levels of generality in graph theory, geometry, and topology. A combinatorial tree (i.e., a connected circuit-free graph) determines an integer-valued metric on its set of vertices; the resulting metric space is called a $\mathbb{Z}$-*tree*. A metric space that is geodesic and uniquely arcwise-connected is called an $\mathbb{R}$-*tree*. These examples can be generalized by defining the notion of a Λ-*tree* [1, Ch. 2, §1], where Λ is any totally ordered Abelian group. A

Λ-tree (X, d) is a Λ-metric space (i.e., its metric takes values in Λ instead of $\mathbb{R}$) satisfying certain natural conditions reflecting the tree-like structure of X.

All isometries of a Λ-tree (X, d) onto itself can be divided into three types: elliptic, hyperbolic, and inversions [1, Ch. 3, §1]. Let g be an isometry of a Λ-tree (X, d). It is called *elliptic* if it has a fixed point in X; g is called an *inversion* if g has no fixed points in X but g^2 does; otherwise g is called *hyperbolic*. The *translation length* of g [5, p. 297] is defined as

$$\|g\| := \begin{cases} 0 & \text{if } g \text{ is an inversion,} \\ \min\{d(x, gx) \colon x \in X\} & \text{otherwise.} \end{cases} \tag{1}$$

In fact, if g is not an inversion, the set of points for which the minimum in (1) is reached is a nonempty closed subtree of X. Hyperbolic isometries are precisely those with $\|g\| > 0$. If g is hyperbolic, then the set $\{x \in X \colon d(x, gx) = \|g\|\}$ is called the *axis* of g; it is isometric to a convex subset of Λ and the action of g on its axis corresponds to the translation by $\|g\|$, which justifies the terminology.

Parry [5] proved that a function $\|\cdot\| \colon G \to \Lambda_+$ is the translation length function for some action of a group G on a Λ-tree if and only if it satisfies a certain set of algebraic conditions; such a function is called a *pseudo-length* on G.

Our main result concerns an explicit formula for $\|g\|$, $g \in \langle a, b\rangle$, in the case of a pair $(a, b) \in G \times G$ satisfying the conditions

$$\|a\| > 0, \quad \|b\| > 0, \quad |\|a\| - \|b\|| < \min\{\|ab\|, \|ab^{-1}\|\}. \tag{2}$$

We call such a pair $(a, b) \in G \times G$ a *ping-pong pair.*

Theorem 1. *If $\|\cdot\|$ is a pseudo-length on a group G and $a, b \in G$ satisfy (2), then*

$$2\|w\| = \left(\sum_{i=1}^{n-1} \|x_i x_{i+1}\|\right) + \|x_n x_1\| > 0,$$

for any cyclically reduced word $w = x_1 \dots x_n$, $x_i \in \{a, b, a^{-1}, b^{-1}\}$, $n \geq 1$.

An important consequence of Theorem 1 is the fact that if G acts by isometries on a Λ-tree (X, d) with the translation length function $\|\cdot\|$, and $(a, b) \in G \times G$ is a ping-pong pair with respect to $\|\cdot\|$, then the subgroup $\langle a, b\rangle \leq G$ is free of rank two and acts freely, without inversions, and properly discontinuously on (X, d). This result is known, see [2, Propositions 1 and 2]. The cited proofs are geometric in nature and rely on drawing pictures or "ping-pong" type arguments. We present a combinatorial approach, using only the defining conditions of a pseudo-length and not referring to any geometric interpretation.

Our other result is the existence and uniqueness of a pseudo-length $\|\cdot\| \colon F(a, b) \to \Lambda_+$ on the free group $F(a, b)$ under certain conditions imposed on the values it takes at a, b, ab, and ab^{-1}.

Theorem 2. *Let $\alpha, \beta, \gamma, \delta \in \Lambda$ be such that*

$$\begin{gathered} \gamma - \alpha - \beta \in 2\Lambda, \quad \delta - \alpha - \beta \in 2\Lambda; \\ \textit{either } \gamma = \delta > \alpha + \beta \quad \textit{or } \max\{\gamma, \delta\} = \alpha + \beta; \\ \alpha > 0,\ \beta > 0,\ |\alpha - \beta| < \min\{\gamma, \delta\}. \end{gathered} \tag{3}$$

There exists exactly one pseudo-length $\|\cdot\| \colon F(a, b) \to \Lambda_+$ such that $\|a\| = \alpha$, $\|b\| = \beta$, $\|ab\| = \gamma$, and $\|ab^{-1}\| = \delta$.

Finally, we use Theorem 2 to prove that, in the case of a subgroup $\Lambda \leq \mathbb{R}$, any *purely hyperbolic* (i.e., $\|g\| > 0$ for $g \neq 1$) pseudo-length on $F(a, b)$ can be described by four elements of Λ satisfying (3), and an outer automorphism of $F(a, b)$. We present an algorithm to effectively find such a description of a given purely hyperbolic pseudo-length on $F(a,b)$. The space of all these pseudo-lengths is related to the concept of the Culler–Vogtmann *outer space* [3].

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The structure of gradient flows with an internal saddle connection on the sphere with holes

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We consider all possible topological structures of typical one–parameter bifurcations of gradient flows on the 2-sphere with holes when number of singular points is at most six. Such flows were completely researched in [2, 3, 4].

Complete topological invariants can be established for the topological equivalence of flows. Usually it is a graph which is augmented with additional information. The purpose of the considered article is to construct a complete invariant for Morse flows and gradient codimension one gradient flows on the 2-sphere with holes which resembles a chord diagram for Morse flows, researched in [1]. This invariant has a marked point in the diagram which allows us to define clearly a number code of the flow. The invariants we have constructed (the distinguishing graph and the flow code) are generalizations of the distinguishing graph of Peixoto and the Oshemkov–Sharko code which had been developed for Morse flows on closed surfaces.

For example, we can consider gradient flows on 2-disk which have only one internal saddle connection whereas all their singular points are hyperbolic.

All possible separatrix connections on a two-dimensional disk with no more than 6 singular points are depicted in Fig. 0.1. The diagrams of reverse flows can be obtained from these by changing all directions and colors of the separatrices (green and red swap places). With such a substitution, diagrams 6 and 10 will revert to themselves (they define one flow each). The remaining diagrams define two flows each.

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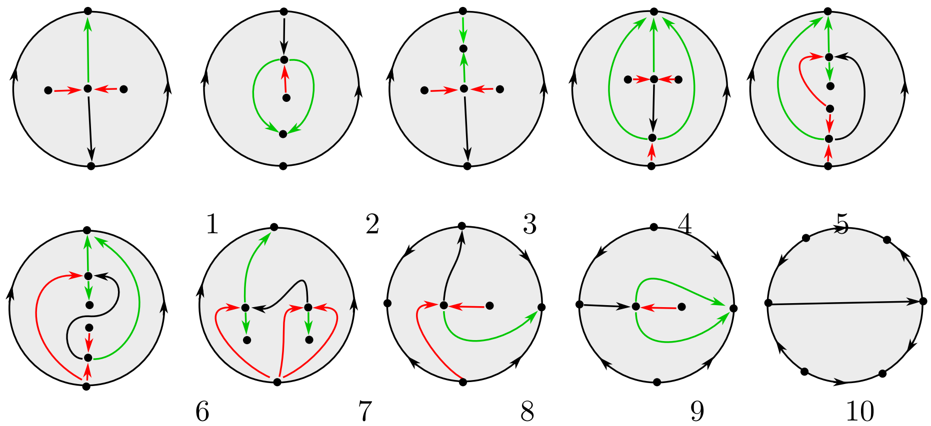

FIGURE 0.1. Saddle connections on D^2

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C^∞-structures approach to travelling wave solutions of the gKdV equation

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In this work, we present a geometric approach to finding travelling wave solutions of the generalized Korteweg–de Vries (gKdV) equation [1]. The method is rooted in the theory of $\mathcal{C}^\infty$-structures and leverages concepts from differential geometry, particularly the theory of distributions and Pfaffian systems (see [2]).

The gKdV equation, given by

$$u_t + u_{xxx} + a(u)u_x = 0, \tag{1}$$

where $u = u(t,x)$ and $a(u)$ is a smooth function, is an important equation in mathematics and physics, with the standard Korteweg–de Vries equation being a well-known special case. The function $a(u)$ plays a crucial role in determining the specific characteristics of the gKdV equation and its applicability to different physical scenarios, by establishing the nature and strength of the nonlinearity in the system.

The core of this talk lies in the application of the C^∞-structure-based method to integrate the ordinary differential equation (ODE) obtained from the travelling wave ansatz applied to (1), i.e., the equation

$$-cy' + y''' + a(y)y' = 0, \tag{2}$$

where $y = y(z)$ and $c \in \mathbb{R}$.

Roughly speaking, a $\mathcal{C}^\infty$-structure for an mth-order ODE is an ordered collection of m vector fields giving rise to a sequence of involutive distributions, starting with the distribution generated by the vector field associated to the ODE. The key geometric insight is that the integral manifolds of these distributions contain the prolongation of the solutions of the ODE. The method enables the integration by transforming the problem into a sequence of m completely integrable Pfaffian equations.

To apply this geometric method to equation (2), we first construct a $\mathcal{C}^\infty$-structure for this ODE, starting with the Lie symmetry ∂_z of the equation. The $\mathcal{C}^\infty$-structure-based integration algorithm is then applied, leading to a sequence of three Pfaffian equations. At the final step we obtain an implicit general solution for the travelling wave solutions of the gKdV equation.

Finally, we present explicit solutions for the gKdV equation in various cases, depending on the choice of the function $a(u)$: the modified KdV equation, a family of KdV equations with power-law nonlinearity, and the Schamel–Korteweg–de Vries equation.

This is a joint work with Concepción Muriel and Adrián Ruiz. This talk is supported by *Plan Propio de estímulo y apoyo a la Investigación y Transferencia 2025-2027, Universidad de Cádiz.*

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State space of compact quantum groups

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Abstract

Green's relations are of fundamental importance in semigroup theory, as they classify the elements of a semigroup based on the principal ideals they generate. In this presentation, we characterize all Green's equivalence relations on the semigroup of states for the classical compact group and cocommutative compact quantum group, and study the structure of all maximal subgroups within this semigroup. We then extend our analysis to the setting of an arbitrary compact quantum group, characterizing all invertible elements in the associated semigroup. Finally, we investigate the semigroup $S(\mathbb{G}_{\mathbb{G}/\mathbb{H}})$ associated with a normal quantum subgroup $\mathbb{H}$ of a compact quantum group $\mathbb{G} = (C(\mathbb{G}), \Delta)$. This work is a collaboration with Prof. Issan Patri and Dr. Malay Mandal.

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On E-endomorphisms of the upper subsemigroup of the bicyclic monoid

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We shall follow the terminology of [1]. By ω we denote the set of all non-negative integers.

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq=1$. Any element of $\mathscr{C}(p,q)$ has the unique representation b^ia^j, $i,j\in\omega$. In [3] the following anti-isomorphic subsemigroups of the bicyclic monoid

$$\mathscr{C}_+(a,b)=\left\{b^ia^j\in\mathscr{C}(a,b)\colon i\leqslant j,\ i,j\in\omega\right\}$$

and

$$\mathscr{C}_-(a,b)=\left\{b^ia^j\in\mathscr{C}(a,b)\colon i\geqslant j,\ i,j\in\omega\right\}$$

are studied. All injective endomorphism of $\mathscr{C}_+(a,b)$ are described in [2].

Let S be a semigroup with the non-empty set of idempotent $E(S)$. An endomorphism α of S is said to be *E-endomorphism* if $(s)\alpha\in E(S)$ for all $s\in S$.

Theorem 1. *Let α be a monoid endomorphism of the semigroup $\mathscr{C}_+(a,b)$. Then the following conditions are equivalent:*

1. *α is an E-endomorphism;*
2. *there exists a non-idempotent element b^ia^j of $\mathscr{C}_+(a,b)$ such that $(b^ia^j)\alpha$ is an idempotent of $\mathscr{C}_+(a,b)$;*
3. *the image $(\mathscr{C}_+(a,b))\alpha$ is a finite subset of $\mathscr{C}_+(a,b)$.*

By $\omega_{\max}$ we denote the set ω with the semilattice operation $n\cdot m=\max\{n,m\}$, $n,m\in\omega$. We extend the semilattice operation of $\omega_{\max}$ onto $\omega^*=\omega\cup\{\infty\}$ with $\infty\notin\omega$ in the following way

$$n\cdot\infty=\infty\cdot n=\infty\cdot\infty=\infty,\qquad \text{for all}\quad n\in\omega.$$

The set ω^* with so defined semilattice operation we denote by $\omega^*_{\max}$.

An endomorphism ε of the semilattice $\omega^*_{\max}$ is called *bounded* if there exists $n_\varepsilon\in\omega$ such that $(x)\varepsilon\leqslant n_\varepsilon$ for all $x\in\omega^*_{\max}$. It is obvious that the composition of any two bounded endomorphisms of the semilattice $\omega^*_{\max}$ is a bounded endomorphism. By $\mathfrak{End}_{\mathbf{b}}(\omega^*_{\max})$ we denote the semigroup of all bounded endomorphisms of semilattice $\omega^*_{\max}$ and by $\mathfrak{End}_{\mathbf{E}}(\mathscr{C}_+(a,b))$ the semigroup of E-endomorphisms of $\mathscr{C}_+(a,b)$.

Theorem 2. *The semigroups $\mathfrak{End}_{\mathbf{b}}(\omega^*_{\max})$ and $\mathfrak{End}_{\mathbf{E}}(\mathscr{C}_+(a,b))$ are isomorphic.*

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On existence of continuations for different types of metrics

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The problems of continuation of a partially defined metric and a partially defined ultrametric were considered in [1] and [2], respectively. Using the language of graph theory we generalize the criteria of existence of continuation obtained in these papers. For these purposes we use the concept of a triangle function introduced by M. Bessenyei and Z. Páles in [3], which gives a generalization of the triangle inequality in metric spaces. The obtained result allows us to get criteria of the existence of continuation for a wide class of semimetrics including metrics, ultrametrics, semimetrics with power triangle inequality, etc.

Let X be a nonempty set. Recall that a mapping $d\colon X\times X\to\mathbb{R}^+$, $\mathbb{R}^+=[0,\infty)$, is a *metric* if for all $x,y,z\in X$ the following axioms hold: (i) $(d(x,y)=0)\Leftrightarrow(x=y)$, (ii) $d(x,y)=d(y,x)$, (iii) $d(x,y)\leqslant d(x,z)+d(z,y)$. The pair (X,d) is called a *metric space*. If only axioms (i) and (ii) hold then the pair (X,d) is called a *semimetric space*. We shall say that d is a *pseudosemimetric* if only axiom (ii) and condition $d(x,x)=0$ hold. In this case the pair (X,d) will be called a *pseudosemimetric space*.

Definition 1. ([3]) Consider a pseudosemimetric space (X,d). We shall say that $\Phi\colon\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is a *triangle function* for d if Φ is symmetric and monotone increasing in both of its arguments, satisfies $\Phi(0,0)=0$ and, for all $x,y,z\in X$, the generalized triangle inequality

$$d(x,y)\leqslant\Phi(d(x,z),d(y,z))$$

holds. We also shall say that d is a Φ-pseudosemimetric if Φ is a triangle function for d.

Let $n\in\mathbb{N}$. For every triangle function Φ consider a function $\Phi^*\colon\mathbb{R}^n_+\to\mathbb{R}^+$ of n variables, defined as

$$\Phi^*(x_1,...,x_n)=\begin{cases}x_1, & \text{if } n=1,\\ \Phi(x_1,x_2), & \text{if } n=2,\\ \Phi(x_1,\Phi(x_2,\Phi(x_3,...\Phi(x_{n-2},\Phi(x_{n-1},x_n))))), & \text{if } n\geqslant 3.\end{cases}$$

It is clear that Φ^* is monotone increasing in all of its variables as well as Φ.

Recall that a graph G is an ordered pair (V,E) consisting of a set $V=V(G)$ of vertices and a set $E=E(G)$ of edges. A graph $G=(V,E)$ together with a weight $w\colon E(G)\to\mathbb{R}^+$ is called a weighted graph. Let (G,w) be a weighted graph and let u,v be vertices belonging to a connected component of G. Let us denote by $\mathcal{P}_{u,v}=\mathcal{P}_{u,v}(G)$ the set of all paths joining u and v in G. For the path $P\in\mathcal{P}_{u,v}$ define the Φ-*weight* of this path by

$$w_\Phi(P)=\begin{cases}0, & \text{if } E(P)=\varnothing,\\ \Phi^*(w(e_1),...,w(e_n)), & \text{otherwise,}\end{cases}$$

where $e_1,...,e_n$ are all edges of the path P. Write

$$d^w_\Phi(u,v)=\inf\{w_\Phi(P)\colon P\in\mathcal{P}_{u,v}\}.$$

In the case $\Phi(x,y)=x+y$ for the connected graph G the function d^w_Φ is a shortest-path pseudometric [1] on the set $V(G)$ and in the case $\Phi(x,y)=\max\{x,y\}$ it is a subdominant pseudoultrametric [2].

In the next lemma and further we identify a pseudosemimetric space (X, d) with the complete weighted graph $(G, w_d) = (G(X), w_d)$ having $V(G) = X$ and satisfying the equality

$$w_d(\{x, y\}) = d(x, y)$$

for every pair of different points $x, y \in X$.

Lemma 2. ([4]) *Let (X, d) be a pseudosemimetric space with the triangle function Φ. Then for every cycle $C \subseteq G(X)$ and for every $e \in E(C)$ the inequality $w_d(e) \leqslant w_\Phi(C \setminus e)$ holds, where $C \setminus e$ is a path obtained from the cycle C by the removal of the edge e.*

We are interested in the following question. Let (G, w) be a weighted graph. Does there exist a Φ-pseudosemimetric $d\colon V(G) \times V(G) \to \mathbb{R}^+$ such that the given weight $w\colon E(G) \to \mathbb{R}^+$ has a continuation to d? I.e., the equality

$$w(\{u, v\}) = d(u, v)$$

holds for all $\{u, v\} \in E(G)$. If such a continuation exists, then we say that w is a *Φ-pseudosemimetrizable weight*.

Theorem 3. ([4]) *Let (G, w) be a weighted graph and let Φ be a continuous in both variables triangle function. The following statements are equivalent.*

- (i) *The weight w is Φ-pseudosemimetrizable.*
- (ii) *The equality $w(\{u, v\}) = d^w_\Phi(u, v)$ holds for all $\{u, v\} \in E(G)$.*
- (iii) *For every cycle $C \subseteq G$ and for every $e \in C$ the inequality $w(e) \leqslant w_\Phi(C \setminus e)$ holds, where $C \setminus e$ is a path obtained from C by the removal of the edge e.*

Corollary 4. ([4]) *Let (G, w) be a weighted graph. Then the corresponding statements are equivalent.*

- (i_1) *The weight w is pseudometrizable, i.e., $\Phi(x, y) = x + y$.*
- (i_2) *For every cycle $C \subseteq G$ the following inequality holds:*

$$2 \max_{e \in E(C)} w(e) \leqslant \sum_{e \in C} w(e).$$

- (ii_1) *The weight w is pseudoultrametrizable, i.e., $\Phi(x, y) = \max\{x, y\}$.*
- (ii_2) *For every cycle $C \subseteq G$ there exist at least two different edges $e_1, e_2 \in E(C)$ such that*

$$w(e_1) = w(e_2) = \max_{e \in E(C)} w(e).$$

- (iii_1) *The weight w is Φ-pseudosemimetrizable with $\Phi(x, y) = (x^p + y^p)^{\frac{1}{p}}$, $p > 0$.*
- (iii_2) *For every cycle $C \subseteq G$ and every $e \in C$ the following inequality holds:*

$$w(e) \leqslant \left(\sum_{\tilde{e} \in C \setminus e} w^p(\tilde{e}) \right)^{\frac{1}{p}}.$$

- (iv_1) *The weight w is Φ-pseudosemimetrizable with $\Phi(x, y) = \varphi^{-1}(\varphi(x) + \varphi(y))$, where $\varphi\colon [0, \infty) \to [0, \infty)$ is a homeomorphism.*
- (iv_2) *For every cycle $C \subseteq G$ and every $e \in C$ the following inequality holds:*

$$w(e) \leqslant \varphi^{-1} \left(\sum_{\tilde{e} \in C \setminus e} \varphi(w(\tilde{e})) \right).$$

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Generalized symmetries of Burgers equation and related algebraic structures

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Although the Burgers equation is the most famous C-integrable model with various applications, its algebra of generalized symmetries has not been exhaustively described despite a number of relevant considerations in the literature. Filling this gap in [3], we presented a basis of this algebra in the most explicit form. We prefered to make a closed and simple proof from scratch, based on relations between the (1+1)D (linear) heat equation $\mathcal{L}_1$, the potential Burgers equation $\mathcal{L}_2$ and the Burgers equation $\mathcal{L}_3$,

$$\mathcal{L}_1\colon\ u_t=u_{xx}\quad\xleftrightarrow{u=\mathrm{e}^w}\quad\mathcal{L}_2\colon\ w_t=w_{xx}+w_x^2\quad\xleftrightarrow{-2w_x=v}\quad\mathcal{L}_3\colon\ v_t+vv_x=v_{xx},$$

which leads to the linearization of $\mathcal{L}_3$ to $\mathcal{L}_1$ by the Hopf–Cole transformation $v=-2u_x/u$. Another important ingredient is the exhaustive description of generalized symmetries of $\mathcal{L}_1$ in [2, Section 6]. The core of the proof is essentially simplified by using the original technique of choosing special coordinates in the associated jet space. Below, instead of the total derivative operators with respect to t and x, we use their restrictions to the solution set of the corresponding equation $\mathcal{L}_i$,

$$\mathrm{D}_x:=\partial_x+\sum_{k=0}^{\infty}z^i_{k+1}\partial_{z^i_k},\quad \mathrm{D}_t:=\partial_t+\sum_{k=0}^{\infty}\big(\mathrm{D}_x^kL^i[z^i]\big)\partial_{z^i_k},$$

where $L^1[u]:=u_{xx}$, $L^2[w]:=w_{xx}+w_x^2$, $L^3[v]:=v_{xx}-vv_x$, $z^i_0:=z^i$, the jet variable z^i_k is identified with the derivative $\partial^kz^i/\partial x^k$, $k\in\mathbb{N}$, $z^1:=u$, $z^2:=w$ and $z^3:=v$.

Recall [1, Section 6] that the algebra of generalized symmetries of the (1+1)-dimensional linear heat equation $\mathcal{L}_1$ is $\Sigma_1=\Lambda_1\Subset\Sigma_1^{-\infty}$, where $\Lambda_1:=\big\langle\mathfrak{Q}^{kl},\,k,l\in\mathbb{N}_0\big\rangle$, $\Sigma_1^{-\infty}:=\big\{\mathfrak{Z}(h)\big\}$ with $\mathfrak{Q}^{kl}:=(\mathrm{G}^k\mathrm{P}^lu)\partial_u$, $\mathrm{P}:=\mathrm{D}_x$, $\mathrm{G}:=t\mathrm{D}_x+\frac12x$, $\mathfrak{Z}(h):=h(t,x)\partial_u$, and the parameter function h runs through the solution set of $\mathcal{L}_1$. Elements of $\Sigma_1^{-\infty}$ are considered trivial generalized symmetries of $\mathcal{L}_1$ since in fact these are Lie symmetries of $\mathcal{L}_1$ that are associated with the linear superposition of solutions of $\mathcal{L}_1$. The complement subalgebra Λ_1 of $\Sigma_1^{-\infty}$ in Σ_1, which is constituted by the linear generalized symmetries of the equation $\mathcal{L}_1$, can be called the essential algebra of generalized symmetries of this equation. The algebra Λ_1 is generated by the two recursion operators P and G from the simplest linear generalized symmetry $u\partial_u$, and both the recursion operators and the seed symmetry are related to Lie symmetries.

Pulling back the elements of the algebra Σ_1 by the transformation $u = \mathrm{e}^w$, we obtain the algebra $\Sigma_2 = \Lambda_2 \Subset \Sigma_2^{-\infty}$ of generalized symmetries of the potential Burgers equation $\mathcal{L}_2$, which is thus isomorphic to the algebra Σ_1. As the counterparts of Λ_1 and $\Sigma_1^{-\infty}$, the subalgebra Λ_2 and the ideal $\Sigma_2^{-\infty}$ of Σ_2 are called the essential and the trivial algebras of generalized symmetries of $\mathcal{L}_2$, respectively.

Theorem 1. *The algebra of generalized symmetries of $\mathcal{L}_3$ is $\Sigma_3 := \big\langle \hat{\mathfrak{Q}}^{kl},\ (k,l) \in \mathbb{N}_0^2 \setminus \{(0,0)\} \big\rangle$ with $\hat{\mathfrak{Q}}^{kl} := \big(\mathrm{D}_x \hat{\mathrm{G}}^k \hat{\mathrm{P}}^l 1\big)\partial_v$, where $\hat{\mathrm{P}} := \mathrm{D}_x - \frac12 v$ and $\hat{\mathrm{G}} := t\mathrm{D}_x + \frac12(x - vt)$.*

Corollary 2. *The homomorphism $\varphi\colon \Lambda_2 \to \Sigma_3$ of the algebra Λ_2 of essential generalized symmetries of the equation $\mathcal{L}_2$ to the entire algebra Σ_3 of generalized symmetries of the equation $\mathcal{L}_3$, which is induced by the differential substitution $-2w_x = v$, is an epimorphism, and $\ker\varphi = \langle \tilde{\mathfrak{Q}}^{00} \rangle$.*

Corollary 3. *The algebra Σ_3 of generalized symmetries of the Burgers equation $\mathcal{L}_3$ is isomorphic to the quotient algebra $A_1(\mathbb{R})^{(-)}/\langle 1 \rangle$, where $A_1(\mathbb{R})^{(-)}$ is the Lie algebra associated with the first Weyl algebra $A_1(\mathbb{R})$, and $\langle 1 \rangle$ is its center. Hence the algebra Σ_3 is simple and two-generated.*

The two-generation of Σ_3 as a Lie algebra means that there is a pair of its elements such that Σ_3 coincides with its subalgebra containing all successive commutators of these two elements. Examples of such pairs are in particular $\{\mathfrak{Q}^{20}, \mathfrak{Q}^{03}\}$ and $\{\mathfrak{Q}^{11},\ \mathfrak{Q}^{10} - \mathfrak{Q}^{02} + \mathfrak{Q}^{30}\}$.

The commutation relations of the algebra Σ_3 are

$$[\hat{\mathfrak{Q}}^{kl}, \hat{\mathfrak{Q}}^{k'l'}] = \sum_{i=1}^{\infty} i! \left(\binom{k'}{i}\binom{l}{i} - \binom{k}{i}\binom{l'}{i} \right) \hat{\mathfrak{Q}}^{k+k'-i,\, l+l'-i},$$

where $(k,l),(k',l') \in \mathbb{N}_0^2 \setminus \{(0,0)\}$, and $\hat{\mathfrak{Q}}^{00} := 0$. In particular, $[\hat{\mathfrak{Q}}^{11}, \hat{\mathfrak{Q}}^{kl}] = (k-l)\hat{\mathfrak{Q}}^{kl}$, i.e., the operator $\mathrm{ad}_{\hat{\mathfrak{Q}}^{11}}$ is a diagonal inner derivation in the basis $(\hat{\mathfrak{Q}}^{kl})$ and the subspace $\Gamma_m := \langle \hat{\mathfrak{Q}}^{kl} \mid k - l = m \rangle$ of Σ_3 is the eigenspace of the operator $\mathrm{ad}_{\hat{\mathfrak{Q}}^{11}}$ corresponding to the eigenvalue m. The Jacobi identity for the Lie bracket implies that $[\Gamma_m, \Gamma_{m'}] \subseteq \Gamma_{m+m'}$ for any $m, m' \in \mathbb{Z}$. As a result, the decomposition of the algebra Σ_3 as the direct sum of its subspaces Γ_m, $\Sigma_3 = \oplus_{m\in\mathbb{Z}}\Gamma_m$, is a $\mathbb{Z}$-grading of this algebra.

Corollary 4. *The space of generalized symmetries of the Burgers equation $\mathcal{L}_3$ that are of order not greater than n is $\Sigma_3^n := \big\langle \hat{\mathfrak{Q}}^{kl},\ (k,l) \in \mathbb{N}_0^2 \setminus \{(0,0)\},\ k + l \leqslant n \big\rangle$, and $\dim \Sigma_3^n = \frac12 n(n+3)$.*

Recall that the maximal Lie invariance algebra $\mathfrak{g}^{\mathrm{B}}$ of $\mathcal{L}_3$ is five-dimensional, $\mathfrak{g}^{\mathrm{B}} = \langle \hat{\mathcal{P}}^t, \hat{\mathcal{D}}, \hat{\mathcal{K}}, \hat{\mathcal{P}}^x, \hat{\mathcal{G}}^x \rangle$, where $\hat{\mathcal{P}}^t = \partial_t$, $\hat{\mathcal{D}} = 2t\partial_t + x\partial_x - v\partial_v$, $\hat{\mathcal{K}} = t^2\partial_t + tx\partial_x + (x - tv)\partial_v$, $\hat{\mathcal{G}}^x = t\partial_x + \partial_v$, $\hat{\mathcal{P}}^x = \partial_x$. In fact, the spaces $\Sigma_3^1 = \big\langle \hat{\mathfrak{Q}}^{01},\ \hat{\mathfrak{Q}}^{10} \big\rangle$ and $\Sigma_3^2 = \big\langle \hat{\mathfrak{Q}}^{01},\ \hat{\mathfrak{Q}}^{10},\ \hat{\mathfrak{Q}}^{02},\ \hat{\mathfrak{Q}}^{11},\ \hat{\mathfrak{Q}}^{20} \big\rangle$ of generalized symmetries of the Burgers equation $\mathcal{L}_3$ that are of order not greater than one and two are closed with respect to Lie bracket of generalized vector fields, i.e., they are a one- and a five-dimensional subalgebras of Σ_3, respectively. They are constituted by the canonical evolution forms of elements of the (nil)radical $\langle \hat{\mathcal{P}}^x, \hat{\mathcal{G}}^x \rangle$ of $\mathfrak{g}^{\mathrm{B}}$ and of the entire algebra $\mathfrak{g}^{\mathrm{B}}$ and thus respectively isomorphic to these algebras. More specifically, the basis elements $\hat{\mathcal{P}}^t$, $\hat{\mathcal{D}}$, $\hat{\mathcal{K}}$, $\hat{\mathcal{G}}^x$ and $\hat{\mathcal{P}}^x$ of $\mathfrak{g}^{\mathrm{B}}$ are associated, up to their signs, with the elements $2\hat{\mathfrak{Q}}^{02}$, $4\hat{\mathfrak{Q}}^{11}$, $2\hat{\mathfrak{Q}}^{20}$, $2\hat{\mathfrak{Q}}^{10}$ and $2\hat{\mathfrak{Q}}^{01}$ of Σ_3^2, respectively.

The algebra Σ_3^1 is the only finite-dimensional maximal Abelian subalgebra of Σ_3. We conjecture that the algebra Σ_3^2 is the only finite-dimensional maximal subalgebra of Σ_3.

Using the results of [1], we can describe maximal Abelian subalgebras of Σ_3. Each of the other maximal Abelian subalgebras of Σ_3 is infinite-dimensional since it contains a subalgebra of the form $\langle (\mathrm{D}_x \mathrm{Q}^k 1)\partial_u,\ k \in \mathbb{N} \rangle$ with a nonconstant polynomial Q of $\hat{\mathrm{G}}$ and $\hat{\mathrm{P}}$. Moreover, it coincides with the centralizer of an element of $\Sigma_3 \setminus \langle \hat{\mathfrak{Q}}^{01}, \hat{\mathfrak{Q}}^{10} \rangle$, which necessarily belongs to it, or, equivalently, with the centralizer of any element of its relative complement to $\langle \hat{\mathfrak{Q}}^{01}, \hat{\mathfrak{Q}}^{10} \rangle$.

We also show that the two well-known recursion operators of the Burgers equation and its two seed generalized symmetries, which are evolution forms of its Lie symmetries, suffice to generate this algebra within the framework of the formal approach, whereas the zero generalized symmetry is sufficient as the only seed symmetry if the recursion operators are interpreted as Bäcklund transformations for the corresponding tangent bundle.

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Topological, metric and fractal analysis of infinite Bernoulli convolutions governed by convergent positive series

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An *infinite Bernoulli convolution* governed by an absolutely convergent series $r_0 = \sum\limits_{n=1}^{\infty} u_n$ is defined as the distribution of the random variable $\zeta = \sum\limits_{n=1}^{\infty} \zeta_n u_n$, where (ζ_n) is a sequence of independent random variables taking values 0 and 1 with probabilities p_0 and $p_1 = 1 - p_0$, respectively.

A *generalized infinite Bernoulli convolution* is defined as the distribution of the random variable $\xi = \sum\limits_{n=1}^{\infty} \xi_n u_n$, where (ξ_n) is a sequence of independent random variables taking values in $\{0, 1, ..., r\}$ with respective probabilities $p_0, p_1, ..., p_r$. The main object of study in this report is the distribution of the random variable

$$\xi = \sum_{n=1}^{\infty} \frac{\xi_n}{s^n} = \Delta^{r_s}_{\xi_1 \xi_2 \ldots \xi_n \ldots},$$

where s and r are natural parameters such that $1 < s \leq r$.

The set E_ξ of values of the random variable ξ is the segment $[0, \frac{r}{s-1}]$.

According to the Jessen–Wintner theorem, the distribution of a random variable is either purely discrete, purely singular, or purely absolutely continuous. We are particularly interested in the conditions under which the distribution is concentrated on a set of Lebesgue measure zero and in determining the fractal dimension of such a set.

The main difficulties in obtaining a complete answer to these questions arise from the ambiguity of number representations in the numeral system with base s and the redundant alphabet $A = \{0, 1, ..., r\}$.

The report is devoted to the case $s=3=r$. From here on, we write

$$\Delta^{r_s}_{\alpha_1\alpha_2\ldots\alpha_k\ldots} = \Delta_{\alpha_1\alpha_2\ldots\alpha_k\ldots} = \sum_{n=1}^{\infty} 3^{-n}\alpha_n, \quad \alpha_n \in A \equiv \{0,1,2,3\}.$$

Lemma 1. *The set* $C[\Delta;\{0,1,3\}] = \{x : \ x = \Delta_{\alpha_1\alpha_2\ldots\alpha_n\ldots}, \alpha_n \in \{0,1,3\}\}$ *is a nowhere dense, N-self-similar set of zero Lebesgue measure, whose Hausdorff dimension is equal to* $\log_3 \frac{3+\sqrt{5}}{2}$.

Theorem 2. *Let* $p_0p_1p_2p_3 = 0$.

1. *If* $p_i = p_{i+1} = p_{i+2} = \frac{1}{3}$, *then* ξ *has a* uniform distribution *on the unit interval.*
2. *If there exists* p_j *such that* $0 \neq p_j \neq \frac{1}{3}$, *then the distribution of the random variable* ξ *s* singular, *and:*

   2.1) if two of the probabilities are zero, then ξ *has a* Cantor-type distribution *with spectrum of fractal dimension* $\log_3 2$*;*

   2.2) if exactly one is zero and $p_1p_2 = 0$, *then* ξ *has a* Cantor-type distribution *with spectrum of fractal dimension* $\log_3 \frac{3+\sqrt{5}}{2}$*;*

   2.3) if exactly one is zero and $p_0p_3 = 0$, *then* ξ *has a* singular distribution with a strictly increasing distribution function, *and the fractal dimension of the essential support of its density is* $-\log_3 p_1^{p_1}p_2^{p_2}p_i^{p_i}$.

Theorem 3. *If* $p_1 = \frac{1}{3} = p_2$, *then the random variable* $\xi = \Delta_{\xi_1\xi_2\ldots\xi_n\ldots}$ *with independent digits* $\xi_n \in \{0,1,2,3\}$ *having probabilities* p_0, p_1, p_2, p_3 *has an absolutely continuous distribution. Moreover, the distribution of* ξ *is the convolution of the uniform distribution on* $[0;1]$ *and a singular Cantor-type distribution. In all other cases, the distribution of* ξ *is singular.*

Corollary 4. *Every infinite Bernoulli convolution governed by the series*

$$\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^2}+\frac{1}{3^2}+\ldots+\frac{1}{3^n}+\frac{1}{3^n}+\frac{1}{3^n}+\ldots,$$

has a purely singular distribution.

Lemma 5. *The sum of two independent singularly distributed random variables*

$$\theta = \sum_{n=1}^{\infty}\theta_n 3^{-n} \text{ and } \varepsilon = \sum_{n=1}^{\infty}\varepsilon_n 3^{-n},$$

where (θ_n) *and* (ε_n) *are sequences of independent random variables taking values in* $\{0,2\}$ *and* $\{0,1\}$, *respectively, with probabilities* $u, 1-u$ *and* $v, 1-v$, *has a singular distribution whose spectrum is the interval* $[0;\frac{3}{2}]$.

Theorem 6. *If* $p_0 = (p_0+p_1)(p_0+p_2)$ *then the distribution of* ξ *is the convolution of two Cantor-type distributions, namely, the distributions of the random variables*

$$\theta = \Delta_{\theta_1\theta_2\ldots\theta_n\ldots}, \quad \varepsilon = \Delta_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots},$$

where (θ_n) *and* (ε_n) *are sequences of independent random variables taking values in* $\{0,2\}$ *and* $\{0,1\}$, *with probabilities* p_0+p_1 *and* $1-(p_0+p_1)$, *and* p_0+p_2 *and* $1-(p_0+p_2)$, *respectively.*

Remark 7. The proof of Theorem 3 is based on the method of characteristic functions (integral transforms) and the method of extracting the absolutely continuous component.

Remark 8. The problem of describing the topological, metric, and fractal properties of the essential support of the density

$$N_\xi = \{x : F'_\xi(x) > 0 \text{ або } F'_\xi(x) \text{ does not exist}\}$$

of the distribution ξ under the condition $p_0p_1p_2p_3 \neq 0$, remains open.

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Topological structure of pre-Hamiltonian flows on the projective plane

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Definition 1. *Pre-Hamiltonian flow* on the projective plane $\mathbb{R}P^2$ is a flow flows whose lift to the double cover (S^2) is a Hamiltonian flow on S^2 with a Hamiltonian that is a Morse function. A flow is *simple* if there are no saddle connections between different saddles within it. *Topological equivalence* of flows is a homeomorphism of the surface that maps trajectories to trajectories and preserve their direction.

For the topological classification of pre-Hamiltonian flows on $\mathbb{R}P^2$, we construct a complete topological invariant of the flow – a distinguishing graph. This invariant is a rooted oreinted tree and is a Reeb graph. In this case, Hamiltonian flows are divided into two types: 1) those whose root of the distinguishing graph has degree 1, and 2) those whose root has degree 2. All other vertices have degree 1 or 3.

Theorem 2. *Two simple prohamiltonian flows on the projective plane are topologically equivalent if and only if their distinguishing graphs are equivalent*

The presence of a marked vertex (root) in the distinguishing graph allows for the efficient computation of the number of topologically non-equivalent graphs with a given number of saddles.

Theorem 3. *The number of topologically non-equivalent simple pre-Hamiltonian flows with k saddles on the projective plane $\mathbb{R}P^2$ can be calculated using the formula*

$$N(\mathbb{R}P^2)_k = K_k + \sum_{i=0}^{k-1} K_i K_{k-i-1},$$

where

$$K_{2n} = 3(K_0K_{2n-1} + K_1K_{2n-2} + \ldots + K_{n-1}K_n),$$

$$K_{2n+1} = 3(K_0K_{2n} + K_1K_{2n-1} + \ldots + K_{n-1}K_{n+1}) + \frac{3K_n^2 + K_n}{2}.$$

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Directional Maximal Operators and Kakeya-Type Sets

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A well-known result of the theory of differentiation of integrals is the *Lebesgue Differentiation Theorem*. This theorem states that for any integrable function $f \in L^1(\mathbb{R}^n)$, for almost every point $x \in \mathbb{R}^n$, the average value of $|f|$ over balls centered at x converges to $f(x)$ when the radius of these balls shrinks to zero. This important result is a consequence of the weak-type boundedness of the Hardy-Littlewood Maximal Operator in L^p spaces.

Naturally, one might ask whether this result remains true if we consider averages over other types of subsets, such as a collection of rectangles assigned to a set of directions.

In this talk, we will discuss a recent result that provides a condition on a set of directions $\Omega \subseteq \mathbb{S}^1$ sufficient to show the admissibility of Kakeya-type sets, extending prior work of Bateman and Katz. This condition guarantees that the associated directional maximal operator M_Ω is unbounded on $L^p(\mathbb{R}^2)$ for every $1 \leq p < \infty$.

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On lower distance estimates for one class of homeomorphisms

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Everywhere further, (X, d, μ) and (X', d', μ') are metric spaces with metrics d and d' and locally finite Borel measures μ and μ', correspondingly. Let G and G' be domains with finite Hausdorff dimensions α and $\alpha' \geqslant 2$ in (X, d, μ) and (X', d', μ'), respectively. For $x_0 \in X$ and $r > 0$, $B(x_0, r)$ and $S(x_0, r)$ denote the ball $\{x \in X : d(x, x_0) < r\}$ and the sphere $\{x \in X : d(x, x_0) = r\}$, correspondingly. Put

$$d(E) := \sup_{x,y\in E} d(x, y)\,.$$

Given $0 < r_1 < r_2 < \infty$, denote $A = A(x_0, r_1, r_2) = \{x \in X : r_1 < d(x, x_0) < r_2\}$. Let $p \geqslant 1$ and $q \geqslant 1$, and let $Q : G \to [0, \infty]$ be a measurable function. Similarly to [1, Ch. 7], a homeomorphism $f : G \to G'$ is called a *ring Q-homeomorphism at a point $x_0 \in \overline{G}$ with respect to (p, q)-moduli*, if the inequality

$$M_p(f(\Gamma(S(x_0, r_1), S(x_0, r_2), A(x_0, r_1, r_2)))) \leqslant \int\limits_{A(x_0,r_1,r_2)\cap G} Q(x) \cdot \eta^q(d(x, x_0))\, d\mu(x) \tag{1}$$

holds for all $0 < r_1 < r_2 < r_0 := d(G)$ and each measurable function $\eta : (r_1, r_2) \to [0, \infty]$ with

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1\ . \tag{2}$$

We say that $f : G \to G'$ is a *ring Q-homeomorphism at a point* $x_0 \in \overline{G}$, if the latter is true for $p = \alpha'$ and $q = \alpha$. For $X = X' = \mathbb{R}^n$, $n \geqslant 2$, we set $d(x, y) = d'(x, y) = |x - y|$, and $\mu = \mu' = m$, where m is the Lebesgue measure. Due to [2], a domain D in $\mathbb{R}^n$ is called a *quasiextremal distance domain* (a *QED-domain for short)* if

$$M(\Gamma(E, F, \mathbb{R}^n)) \leqslant A \cdot M(\Gamma(E, F, D)) \tag{3}$$

for some finite number $A \geqslant 1$ and all continua E and F in D. In the same way, one can define quasiextremal distance domains in an arbitrary metric measure space.

Given a compact set K in a domain D, we set $d(K, \partial D) = \inf\limits_{x\in K, y\in\partial D} d(x, y)$. If $\partial D = \varnothing$, we set $d(K, \partial D) = \infty$.

Given a domain D in $\mathbb{R}^n$, $n \geqslant 2$, a Lebesgue measurable function $Q : D \to [0, \infty]$, a compact set $K \subset D$ and numbers $A \geqslant 1, \delta > 0$ denote by $\mathfrak{F}^{A,\delta}_{K,Q}(D)$ a family of all mappings $f : D \to \mathbb{R}^n$ satisfying the relations (1)–(2) at any point $x_0 \in D$ with $d(x, y) = d'(x, y) = |x-y|$ and $\mu = \mu' = m$, where m is the Lebesgue measure, such that $D_f := f(D)$ is a QED-domain with A in (3) and, in addition, $d(f(K), \partial D_f) \geqslant \delta$. The following result holds.

Theorem 1. *If $Q \in L^1(D)$, then there exist constants $C, C_1 > 0$ such that*

$$|f(x) - f(y)| \geqslant C_1 \cdot \exp\left\{-\frac{\|Q\|_1 A}{C|x-y|^n}\right\} \tag{4}$$

for all $x, y \in K$ and every $f \in \mathfrak{F}^{A,\delta}_{K,Q}(D)$.

Theorem 1 admits an analogue in metric spaces, which we will now formulate.

Let X be a metric space. We say that the condition of the *complete divergence of paths* is satisfied in $D \subset X$ if for any different points y_1 and $y_2 \in D$ there are some w_1, $w_2 \in \partial D$ and paths $\alpha_2 : (-2, -1] \to D$, $\alpha_1 : [1, 2) \to D$ such that 1) α_1 and α_2 are subpaths of some geodesic path $\alpha : [-2, 2] \to X$, that is, $\alpha_2 := \alpha|_{(-2,-1]}$ and $\alpha_1 := \alpha|_{[1,2)}$; 2) 2) the geodesic path α joins the points w_2, y_2, y_1 and w_1 such that $\alpha(-2) = w_2$, $\alpha(-1) = y_2$, $\alpha(1) = y_1$, $\alpha(2) = w_1$.

Note that the condition of the complete divergence of the paths is satisfied for an arbitrary bounded domain D' of the Euclidean space $\mathbb{R}^n$. Let (X, μ) be a metric space with measure μ and of Hausdorff dimension n. For each real number $n \geqslant 1$, we define *the Loewner function* $\Phi_n : (0, \infty) \to [0, \infty)$ on X as

$$\Phi_n(t) = \inf\{M_n(\Gamma(E, F, X)) : \Delta(E, F) \leqslant t\}\,, \tag{5}$$

where the infimum is taken over all disjoint nondegenerate continua E and F in X and

$$\Delta(E, F) := \frac{\mathrm{dist}\,(E, F)}{\min\{d(E), d(F)\}}\,.$$

A pathwise connected metric measure space (X, μ) is said to be a *Loewner space* of exponent n, or an n-Loewner space, if the Loewner function $\Phi_n(t)$ is positive for all $t > 0$ (see [1, Section 2.5] or [3, Ch. 8]). Observe that, $\mathbb{R}^n$ and $\mathbb{B}^n \subset \mathbb{R}^n$ are Loewner spaces (see [3, Theorem 8.2 and Example 8.24(a)]).

Given a domain D in X, $n \geqslant 2$, a measurable function $Q : D \to [0, \infty]$, a compact set $K \subset D$ and numbers $A, \delta > 0$ denote by $\mathfrak{F}^{A,\delta}_{K,Q}(D)$ a family of all mappings $f : D \to X'$ satisfying the relations (1)–(2) at any point $x_0 \in D$, such that $D_f := f(D)$ is a compact QED-subdomain of X' with A in (3) and, in addition, $d'(f(K), \partial D_f) \geqslant \delta$. The following result holds.

Theorem 2. *Let (X, d, μ) and (X', d', μ') be metric spaces with metrics d and d′ and locally finite Borel measures μ and μ', correspondingly. Assume that, the condition of the complete divergence of paths is satisfied in a domain $D \subset X$. Let X' be a n-Loewner space in which the relation $\mu(B_R) \leqslant C^*R^n$ holds for some constant $C^* \geqslant 1$, for some exponent $n > 0$ and for all closed balls B_R of radius $R > 0$. If $Q \in L^1(D)$, then there exist constants $C, C_1 > 0$ such that*

$$d'(f(x), f(y)) \geqslant C_1 \cdot \exp\left\{-\frac{\|Q\|_1 A}{C d^n(x, y)}\right\} \tag{6}$$

for all $x, y \in K$ and every $f \in \mathfrak{F}^{A,\delta}_{K,Q}(D)$.

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On floating bodies and related topics

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This is a joint work with Maria Alfonseca, Fedor Nazarov, Alina Stancu and Vlad Yaskin.

Let K be a convex body in $\mathbb{R}^2$. For every $\theta \in \mathbb{R}$ and the corresponding unit vector $e(\theta) = (\cos\theta, \sin\theta)$ and for every $t \in \mathbb{R}$, define the half-planes

$$W^+(\theta, t) = \{x : \langle x, e(\theta)\rangle \geq t\} \quad \text{and} \quad W^-(\theta, t) = \{x : \langle x, e(\theta)\rangle \leq t\}.$$

If $0 < \mathcal{D} < 1$, then for every $\theta \in \mathbb{R}$, there is a unique $t(\theta)$ such that

$$\mathrm{vol}_2(W^+(\theta, t(\theta)) \cap K) = \mathcal{D}\,\mathrm{vol}_2(K).$$

The corresponding convex body of flotation $K^{\mathcal{D}}$ is defined as

$$K^{\mathcal{D}} = \bigcap_{\theta\in\mathbb{R}} W^-(\theta, t(\theta)).$$

We investigate the homothety conjecture for convex bodies of flotation of planar domains. We show that there is a density close to $\frac{1}{2}$ for which there is a body K different from an ellipse with the property that $K^{\mathcal{D}}$ is homothetic to K.

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A Banach space characterization of (sequentially) Ascoli spaces

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The talk is based on my recent article [4].

One of the basic theorems in Analysis is the Ascoli theorem which states that if X is a k-space, then every compact subset of $C_k(X)$ is evenly continuous, see Theorem 3.4.20 in [2]. Being motivated by the Ascoli theorem we introduced and studied in [1] the class of Ascoli spaces. A Tychonoff space X is called an *Ascoli space* if every compact subset $\mathcal{K}$ of $C_k(X)$ is evenly continuous, that is the map $X \times \mathcal{K} \ni (x, f) \mapsto f(x)$ is continuous. In other words, X is Ascoli if and only if the compact-open topology of $C_k(X)$ is Ascoli in the sense of [5, p.45].

Being motivated by the classical notion of c_0-barrelled locally convex spaces we defined in [3] a Tychonoff space X to be *sequentially Ascoli* if every convergent sequence in $C_k(X)$ is equicontinuous. Clearly, every Ascoli space is sequentially Ascoli, but the converse is not true in general.

Let X be a Tychonoff space, and let E'_β be the dual space of a locally convex space E. We shall say that a map $T : X \to E'$ is *almost k-compact* (resp., *almost k-sequential*) if it is weak* continuous and there are a neighborhood U of zero in E and a compact subset (resp., a null sequence) K of $C_k(X)$ such that the family $\{T_E(x,a) : \ a \in U\}$ is contained in the absolutely convex closed hull $\overline{\mathrm{acx}}(K)$ of K. Now we formulate the main result of the talk.

Theorem 1. *For a Tychonoff space X, the following assertions are equivalent:*

(i) *X is an Ascoli* (*resp., sequentially Ascoli*) *space;*

(ii) *for each cardinal Γ, every k-continuous and almost k-compact* (*resp., almost k-sequential*) *map $T : X \to \ell_\infty(\Gamma)$ is continuous;*

(iii) *for each Banach space E, every k-continuous and almost k-compact* (*resp., almost k-sequential*) *map $T : X \to E'_\beta$ is continuous.*

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Liminal $\mathbf{SL_2Z}_p$-representations and odd-th cyclic covers of genus one two-bridge knots

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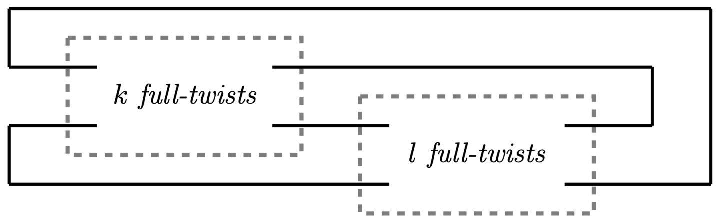
We briefly survey a joint work with Ryoto Tange and Jun Ueki [STU25].

Theorem 1. *It is known that every genus one two-bridge knot is realized as a* double twist knot *of type $J(2k, 2l)$ with $(0,0) \neq (k,l) \in \mathbb{Z}^2$ defined by the following diagram.*

Definition 2. Let p be a prime number. When n ranges over natural numbers, the rings $\mathbb{Z}/p^n\mathbb{Z}$ naturally form an inverse system. The inverse limit of this system is called the ring of p-adic integers and denoted by $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^n\mathbb{Z}$.

Definition 3. Let π be a group.



$J(2k, 2l)$

(1) A function $\chi : \pi \to \mathbb{Z}_p$ is called an $\mathrm{SL}_2\mathbb{Z}_p$-*character* if there exists an SL_2-representation ρ over an extension of $\mathbb{Z}_p$ such that $\chi = \operatorname{tr} \rho$ holds.

(2) An $\mathrm{SL}_2\mathbb{Z}_p$-character is said to be *liminal* if it is absolutely reducible and its every open neighborhood contains an absolutely irreducible $\mathrm{SL}_2\mathbb{Z}_p$-character. Here, a neighborhood refers to a neighborhood with respect to the p-adic distance on the character variety.

Theorem 4. *Let $K = J(2k, 2l)$ be a genus one two-bridge knot in S^3. If a prime number p divides the size of the 1st homology group of some odd-th cyclic branched cover of K, then its group $\pi_1(S^3 - K)$ admits a liminal $\mathrm{SL}_2\mathbb{Z}_p$-character.*

In the proof of this theorem, a nature of certain Lucas-type sequences plays a key role.

Example 5. The sequence (L_n) starting with $L_0 = 2$, $L_1 = 1$, and defined by $L_n = L_{n-1} + L_{n-2}$ is called the Lucas sequence. Calculating from the smallest terms, we get:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571, \mathit{etc.}$$

Focusing on the prime factors of the odd-indexed terms, we get:

$$L_3 = 2^2, L_5 = 11, L_7 = 29, L_9 = 2^2 \times 19, L_{11} = 199, L_{13} = 521, L_{15} = 11 \times 31, \mathit{etc.}$$

The prime factors are either 2, or have a last digit of 1 or 9. This observation can be generalized as follows: If a prime number p divides L_{2n+1} for some $n \in \mathbb{Z}_{\geq 0}$, then $p = 2$ or the Legendre symbol satisfies $\left(\dfrac{5}{p}\right) = 1$.

Proposition 6. *Let $m \in \mathbb{Z}$. Let a and b be the solutions of the equation $t^2 - t + m = 0$, and define $L_n = a^n + b^n$ for any $n \in \mathbb{N}$. If a prime number p divides L_{2n+1} for some $n \in \mathbb{Z}_{\geq 0}$, then the Legendre symbol satisfies*

$$\left(\frac{4m^2 - m}{p}\right) = 1.$$

Proof. Define $F_n = \dfrac{a^n - b^n}{a - b}$ for any $n \in \mathbb{N}$. Then, $F_n \in \mathbb{Z}$, and

$$L_n^2 + (4m - 1)F_n^2 = 4m^n$$

holds. If p divides L_{2n+1} for some $n \in \mathbb{Z}_{\geq 0}$, then

$$(4m - 1)F_{2n+1}^2 \equiv 4m^{2n+1} \bmod p.$$

So, $m(4m - 1)$ is a square mod p. □

Proof of Theorem 4 (Outline). A Seifert matrix of $J(2k, 2l)$ is given by $V = \begin{pmatrix} k & 1 \\ 0 & l \end{pmatrix}$, so the Alexander polynomial is

$$\Delta_{J(2k,2l)}(t) = \det(tV - V^{\perp}) = klt^2 + (1 - 2kl)t + kl.$$

Let α and β denote the solutions $\Delta_K(t) = mt^2 - (1 - 2m)t + m = 0$, where $m = kl$. Then, by using Fox–Weber formula, we can write

$$r_{2n+1} = |\mathrm{Res}(t^n - 1, \Delta_K(t))| = m^{2n+1}(2 - \alpha^{2n+1} - \beta^{2n+1}).$$

Let a and b denote the solutions of $t^2 - t + m = 0$. Then, $\{a^2, b^2\} = \{-m\alpha, -m\beta\}$, and we obtain $r_{2n+1} = L_{2n+1}^2$. Therefore, by **Proposition 6**, if a prime number p divides r_{2n+1} for some $n \in \mathbb{Z}_{\geq 0}$, then the Legendre symbol satisfies $\left(\dfrac{4k^2l^2 - kl}{p}\right) = 1$.

On the other hand, let $S_*(z) \in \mathbb{Z}[z]$ denotes the Chebyshev polynomial of the 2nd kind and define

$$f_{k,l}(x, y) = S_l(z) - (1 + (-x^2 + y + 2)S_{k-1}(y)(S_k(y) - S_{k-1}(y))S_{l-1}(z),$$
$$z = 2 + (y - 2)(-x^2 + y + 2)S_{m-1}^2(y).$$

Then Tran's calculation [Tra18] and Hensel's lemma assure that liminal $\mathrm{SL}_2\mathbb{Z}_p$ characters corresponds to intersection points $(\pm\sqrt{4 - \frac{1}{kl}}, -\frac{1}{kl})$ of the curves $f_{k,l}(x, y) = 0$ and $y - 2 = 0$ in $\mathbb{Z}_p^2$. This completes the proof. $\square$

Remark 7. The analogies between knots and primes, or 3-manifolds and number rings have played important roles since the era of Gauss (cf.[Mor24]). In modern times, among other things, the analogy between the Alexander–Fox theory of $\mathbb{Z}$-covers and the Iwasawa theory of $\mathbb{Z}_p$-extensions of number fields, and that between deformation theories of knot group representations (e.g., Thurston's hyperbolic deformation) and Galois representations (e.g., due to Hida–Mazur) have been pointed out. There are special interests in irreducible $\mathrm{SL}_2\mathbb{Z}_\mathrm{p}$-representations whose residual representations are reducible. In our study [STU25], following Mazur [Maz11], we aimed to "go the other way".

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On fuzzy K-ultrametric spaces

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Let $K \in [0, \infty]$. A metric space (X, d) is called a K-ultrametric space [1, 2] if $d(x, y) \leq K$ whenever $\min\{d(x, z), d(z, y)\} \leq K$. The talk is devoted to a counterpart of this notion in the realm of fuzzy metric spaces [3].

A binary operation $*\colon [0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if it satisfies the following conditions.

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a*1=a$ for all $a\in[0,1]$,
(4) $a*b\le c*d$ whenever $a\le c$ and $b\le d$, for each $a,b,c,d\in[0,1]$.

A triple $(X,M,*)$ is called a fuzzy metric space if X is a nonempty set, $*$ is a continuous t-norm and $M\colon X\times X\times(0,\infty)\to\mathbb{R}$ is a map such that for every $x,y,z\in X$ and $t,s>0$ we have

1) $0<M(x,y,t)\le 1$;
2) $M(x,y,t)=1$ if and only if $x=y$;
3) $M(x,y,t)=M(y,x,t)$;
4) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$;
5) $M(x,y,\cdot)\colon(0,\infty)\to(0,1]$ is continuous.

A triple $(X,M,*)$ is called a fuzzy ultrametric space if X is a nonempty set, $*$ is the minimum $\wedge$ and $M\colon X\times X\times(0,\infty)\to\mathbb{R}$ is a map satisfying conditions 1), 2), 3) and 5) of this definition and moreover

4') $M(x,y,t)*M(y,z,s)\le M(x,z,\max\{t,s\}$ for $x,y,z\in X$ and $t>0$. Condition 4') is equivalent to the condition $M(x,y,t)*M(y,z,t)\le M(x,z,t)$.

Given a nondecreasing function $K\colon(0,\infty)\to[0,1]$, we define a fuzzy K-ultrametric as a fuzzy metric satisfying the condition $M(x,y,t)\wedge M(y,z,t)\le M(x,z,t)$ whenever

$$\max\{M(x,z,t),M(z,y,t)\}\ge K(t).$$

We establish some properties of fuzzy K-ultrametrics and consider questions of fuzzy K-ultrametrization of products, hyperspaces, and spaces of measures on fuzzy K-ultrametric spaces.

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On Koebe-Bloch theorem for mappings with inverse Poletsky inequality

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Let us recall the formulation of the classical Koebe theorem.

Theorem A. *Let $f:\mathbb{D}\to\mathbb{C}$ be an univalent analytic function such that $f(0)=0$ and $f'(0)=1$. Then the image of f covers the open disk centered at 0 of radius one-quarter, that is, $f(\mathbb{D})\supset B(0,1/4)$.*

The main fact contained in the paper is the statement that something similar has been done for a much more general class of spatial mappings. Below $dm(x)$ denotes the element of the Lebesgue measure in $\mathbb{R}^n$. Everywhere further the boundary ∂A of the set A and the closure $\overline{A}$ should be understood in the sense of the extended Euclidean space $\overline{\mathbb{R}^n}$. Recall that, a Borel function $\rho : \mathbb{R}^n \to [0,\infty]$ is called *admissible* for the family Γ of paths γ in $\mathbb{R}^n$, if the relation

$$\int\limits_{\gamma} \rho(x)\,|dx| \geqslant 1$$

holds for all (locally rectifiable) paths $\gamma \in \Gamma$. In this case, we write: $\rho \in \operatorname{adm}\Gamma$. The *modulus* of Γ is defined by the equality

$$M(\Gamma) = \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^n(x)\,dm(x)\,.$$

Let $y_0 \in \mathbb{R}^n$, $0 < r_1 < r_2 < \infty$ and

$$A = A(y_0, r_1, r_2) = \{ y \in \mathbb{R}^n : r_1 < |y - y_0| < r_2 \}\,.$$

Given $x_0 \in \mathbb{R}^n$, we put $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$, $\mathbb{B}^n = B(0,1)$, $S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}$. A mapping $f : D \to \mathbb{R}^n$ is called *discrete* if the pre-image $\{f^{-1}(y)\}$ of any point $y \in \mathbb{R}^n$ consists of isolated points, and *open* if the image of any open set $U \subset D$ is an open set in $\mathbb{R}^n$. Given sets E, $F \subset \overline{\mathbb{R}^n}$ and a domain $D \subset \mathbb{R}^n$ we denote by $\Gamma(E,F,D)$ the family of all paths $\gamma : [a,b] \to \overline{\mathbb{R}^n}$ such that $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for $t \in (a,b)$. Given a mapping $f : D \to \mathbb{R}^n$, a point $y_0 \in \overline{f(D)} \setminus \{\infty\}$, and $0 < r_1 < r_2 < r_0 = \sup\limits_{y \in f(D)} |y - y_0|$, we denote by $\Gamma_f(y_0, r_1, r_2)$ a family of all paths γ in D such that $f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$. Let $Q : \mathbb{R}^n \to [0,\infty]$ be a Lebesgue measurable function. We say that f *satisfies the inverse Poletsky inequality at a point* $y_0 \in \overline{f(D)} \setminus \{\infty\}$ if the relation

$$M(\Gamma_f(y_0, r_1, r_2)) \leqslant \int\limits_{A(y_0,r_1,r_2)\cap f(D)} Q(y)\cdot \eta^n(|y - y_0|)\,dm(y) \tag{1}$$

holds for any Lebesgue measurable function $\eta : (r_1, r_2) \to [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\,dr \geqslant 1\,. \tag{2}$$

The relations (1) are proved for different classes of mappings, see e.g. [2].

Set $q_{y_0}(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{S(y_0,r)} Q(y)\,d\mathcal{H}^{n-1}(y)$, where ω_{n-1} denotes the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. We say that a function $\varphi : D \to \mathbb{R}$ has a *finite mean oscillation* at a point $x_0 \in D$, write $\varphi \in FMO(x_0)$, if $\limsup\limits_{\varepsilon \to 0} \frac{1}{\Omega_n \varepsilon^n} \int\limits_{B(x_0,\,\varepsilon)} |\varphi(x) - \overline{\varphi}_{\varepsilon}|\,dm(x) < \infty$, where $\overline{\varphi}_{\varepsilon} = \frac{1}{\Omega_n \varepsilon^n} \int\limits_{B(x_0,\,\varepsilon)} \varphi(x)\,dm(x)$ and Ω_n is the volume of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$. We also say that a function $\varphi : D \to \mathbb{R}$ has a finite mean oscillation at $A \subset \overline{D}$, write $\varphi \in FMO(A)$, if φ has a finite mean oscillation at any point $x_0 \in A$. Let h be a chordal metric in $\overline{\mathbb{R}^n}$,

$$h(x,\infty) = \frac{1}{\sqrt{1+|x|^2}}\,, \quad h(x,y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}\,, \qquad x \neq \infty \neq y\,,$$

and let $h(E) := \sup\limits_{x,y \in E} h(x,y)$ be a chordal diameter of a set $E \subset \overline{\mathbb{R}^n}$ (see, e.g., [1, Definition 12.1]).

Given a continuum $E \subset D$, $\delta > 0$ and a Lebesgue measurable function $Q : \mathbb{R}^n \to [0, \infty]$ we denote by $\mathfrak{F}_{E,\delta}(D)$ the family of all open discrete mappings $f : D \to \mathbb{R}^n$, $n \geqslant 2$, satisfying relations (1)–(2) at any point $y_0 \in \overline{\mathbb{R}^n}$ such that $h(f(E)) \geqslant \delta$. The following statement holds, cf. [3].

Theorem 1. *Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, and let $B(x_0, \varepsilon_1) \subset D$ for some $\varepsilon_1 > 0$.*

Assume that, $Q \in L^1(\mathbb{R}^n)$ and, in addition, one of the following conditions hold:

1) $Q \in FMO(\overline{\mathbb{R}^n})$;

2) for any $y_0 \in \overline{\mathbb{R}^n}$ there is $\delta(y_0) > 0$ such that

$$\int\limits_0^{\delta(y_0)} \frac{dt}{t q_{y_0}^{\frac{1}{n-1}}(t)} = \infty \,. \tag{3}$$

Then there is $r_0 > 0$, which does not depend on f, such that

$$f(B(x_0, \varepsilon_1)) \supset B_h(f(x_0), r_0) \qquad \forall\ f \in \mathfrak{F}_{E,\delta}(D)\,,$$

where $B_h(f(x_0), r_0) = \{w \in \overline{\mathbb{R}^n} : h(w, f(x_0)) < r_0\}$.

Remark 2. The condition $Q \in FMO(\infty)$ of the condition (3) for $y_0 = \infty$ must be understood as follows: these conditions hold for $y_0 = \infty$ if and only if the function $\widetilde{Q} := Q\left(\frac{y}{|y|^2}\right)$ satisfies similar conditions at the origin.

The result mentioned above is obtained in [4].

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On ruled affine submanifolds with two-dimensional base

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When we study affine submanifolds with flat connection [1, 2, 3] or rank two affine fundamental form [4], it often turns out that representatives of such submanifolds are ruled submanifolds. So, they deserve detailed research.

We consider generalized ruled affine submanifolds, namely ruled affine submanifolds with 2-dimensional base and $n-2$ rulings, in case of codimension 1 and 2. We obtain such a type of ruled affine submanifolds (codimension 2) when we study affine submanifolds of rank two [4].

Detailed description of ruled affine submanifolds of arbitrary dimension and codimension in the classical sense, that is, ruled submanifolds over a curve, can be found in [5].

In case of codimension 1 the base of the ruled affine submanifold is a hyperbolic-type surface in $\mathbb{R}^3$. In case of codimension 2 the base of the ruled affine submanifold is an elliptic-type surface in

$\mathbb{R}^4$. We find conditions on the directions of rulings that follow from the condition of completeness of the affine immersion. In particular, we receive all affine characteristics (induced connection, transversal connection, affine fundamental forms, Weingarten operators, curvature tensor) of such an affine immersion in case the base surface is a complex curve.

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100th anniversary of Sinyukov Mykola Stepanovych (1925-1992)

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On the 4th of July we will mark the 100th anniversary of the date of birth of the outstanding Ukrainian geometrician Sinyukov Mykola Stepanovych (1925 – 1992), doctor of Physical and Mathematical Sciences, professor, professor of Odesa I. I. Mechnikov State University. There are the articles dedicated to the memory of Mykola Sinyukov in the books of abstracts of the International Conferences "Geometry in Odesa – 2010" and "Geometry in Odesa – 2015". The general facts of his biography and his principal scientific achievements are represented there.

On some aspects of vanishing theorems of global character about holomorphically projective mappings of complete Kahlerian spaces

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Generalization of Bochner's technique (see, for example, [1]) allows to extend to noncompact but complete Kahlerian spaces a number of theorems of holomorphically projective unique definability on the whole that have been proved previously only to the compact ones (see, for example, [2]). In particular, the next theorems are true.

Theorem 1. *Complete connected Kahlerian C^r-space K^n $(n > 2,\ r > 2)$ with positive defined metric form and non-negatively defined on the set of symmetric tensors b^{ij} form*

$$T_{\alpha\gamma\sigma\beta}b^{\alpha\beta}b^{\gamma\sigma} \qquad (T_{\alpha\gamma\sigma\beta} = g_{\gamma\beta}R_{\alpha\sigma} - R_{\alpha\gamma\sigma\beta})$$

doesn't admit non-trivial (different from the affine) holomorphically projective mappings on the whole.

Theorem 2. *Complete connected Kahlerian C^r-space K^n ($n > 2$, $r > 4$) with strictly defined form*

$$(2R^{\gamma}_{\alpha,\beta\gamma} - 3R_{\alpha\beta,.\gamma}^{\ \ \ \ \gamma})\eta^{\alpha}\eta^{\beta}$$

doesn't admit non-trivial (different from the affine) holomorphically projective mappings on the whole.

Theorem 3. *Complete connected Kahlerian C^r-space K^n ($n > 2$, $r > 4$) with strictly defined form $R^{i\alpha\ \ \ \beta}_{..\,,\,.\ i}\eta^{\alpha}\eta^{\beta}$ doesn't admit non-trivial (different from the affine) holomorphically projective mappings on the whole.*

Examples of Kahlerian spaces of considered types are pointed out.

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Normal subgroups of iterated wreath products of symmetric groups

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In this research we continue our previous investigation of wreath product normal structure [1, 2]. Normal subgroups and there structures for finite and infinite iterated wreath products $S_{n_1} \wr \ldots \wr S_{n_m}$, $n, m \in \mathbb{N}$ and $A_n \wr S_n$ are founded.

Let $k(\pi)$ be the number of cycles in decomposition of permutation π of degree n. The number $n - k(\pi)$ is denoted by $dec(\pi)$, and is called a decrement [6] of permutation π. If $\pi_1, \pi_2 \in S_n$, then the following formula holds:

$$dec(\pi_1 \cdot \pi_2) = dec(\pi_1) + dec(\pi_2) - 2m, m \in \mathbf{N}, \tag{1}$$

Definition 1. The permutational *subwreath product* $G \wr\wr H$ is the semi-direct product $G \ltimes \tilde{H}^X$, where G acts on the subdirect product [4] $\tilde{H}^X$ by the respective permutations of the subdirect factors. Provided the specification of $\tilde{H}^X$ is established separately.

Definition 2. The set of elements from $S_n \wr S_n, n \geqslant 3$ which presented by the tableaux of form: $[e]_0$, $[a_1, a_2, \ldots, a_n]_1$, satisfying the following condition

$$\sum_{i=1}^{n} dec([a_i]_1) = 2k, k \in \mathbb{N}, \tag{2}$$

be called a generalized alternating group of first level $\widetilde{A}_n^{(1)}$, and denote this set by $E \wr\wr \widetilde{A}_n$. Note that condition (2) uniquely identifies subdirect product.

We spread this definition on 3-multiple wreath product by recursive way.

Definition 3. *The subgroup $E \wr \widetilde{A}_n^{(1)}$ be denoted by $\widetilde{A}_n^{(2)}$.*

Theorem 4. *The subgroup $\widetilde{A}_n^{(1)}$ has **normal rank** $n-1$ [7] in $S_n \wr S_n$, $n \geqslant 3$ provided $n \equiv 1(mod2)$ and **normal rank** n iff $n \equiv 0(mod2)$ and $n \geqslant 3$.*

Theorem 5. *The subgroup $\widetilde{A}_3^{(1)}$ of $S_3 \wr S_3$ has the structure $\widetilde{A}_3^{(1)} \simeq (C_3 \times C_3 \times C_3) \rtimes (C_2 \times C_2)$.*
The structure of subgroup $\widetilde{A}_n^{(1)} \leq S_n \wr S_n$ is $\widetilde{A}_n^{(1)} \simeq (\prod_{i=1}^{n} A_n) \rtimes (\prod_{i=1}^{n-1} C_2)$.

Definition 6. The set of elements from $S_n \wr S_n \wr S_n, n \geqslant 3$ presented by the tables [3] form: $[e]_0$, $[e, e, \ldots, e]_1, [a_1, a_2, \ldots, a_n]_2$, satisfying the following condition

$$\sum_{i=1}^{n} dec([a_i]_2) = 2k, k \in \mathbb{N}, \tag{3}$$

be denoted by $\widetilde{A}_{n^2}^{(2)}$. Note that condition (3) uniquely identifies subdirect product in $\prod_{i=1}^{n^2} S_n$ as base of subwreath product, the similar subdirect product describing commutator of wreath product was investigated by us in [9] in research of pronormality it appears in [8].

Proposition 7. *The subgroup $\widetilde{A}_n^{(1)} \lhd S_n \wr S_n$ as well as $\widetilde{A}_n^{(2)} \lhd S_n \wr S_n \wr S_n$. Furthermore $\widetilde{A}_n^{(2)} \lhd \widetilde{A}_{n^2}^{(2)}$.*

Definition 8. *A subgroup in $S_n \wr S_n$ is called $\widetilde{T_n}$ if it consists of:*

(1) elements of $E \wr A_n$,

(2) elements with the tableau [3] presentation $[e]_1$, $[\pi_1, \ldots, \pi_n]_2$, that $\pi_i \in S_n \setminus A_n$.

One easy can validates a correctness of this definition, i.e. that the set of such elements form a subgroup and its normality. This subgroup has structure

$$\tilde{T_n} \simeq (\underbrace{A_n \times A_n \times \cdots \times A_n}_{n}) \rtimes C_2 \simeq \underbrace{S_n \boxplus S_n \ldots \boxplus S_n}_{n},$$

where the operation $\boxplus$ of a subdirect product is subject of item 1) and 2)

Remark 9. The order of $\widetilde{T_n}$ is $\frac{(n!)^n}{2^{n-1}}$.

Definition 10. The unique minimal normal subgroup is called the monolith.

Theorem 11. *The monolith of $S_n \wr S_m$ is $e \wr A_m$.*

Definition 12. The set of elements from $\overset{k}{\underset{i=1}{\wr}} S_{n_i}, n_i \geqslant 3$ with depth m satisfying the following condition

$$\sum_{i=1}^{n^j} dec([a_i]_j) = 2t, t \in \mathbb{N}, \ m \leq j \leq k, \quad [a_i]_j = e, \, whenever \; j = \overline{1, m-1} \tag{4}$$

be called $\widetilde{A}_{n^j}^{(m,k)}$, where $m < k$.

Theorem 13. *The order of normal subgroup $\widetilde{A}_{n^j}^{(1,k)}$ is $(\frac{1}{2})^k \cdot (n!)^{(\frac{n^{(k+1)}-1}{n-1})}$ and the order of the quotient $\overset{k}{\underset{i=1}{\wr}} S_{n_i} \Big/ \widetilde{A}_{n^j}^{(1,k)}$ is 2^k. The order of generalized alternating group of k-th level $\widetilde{A}_{n^k}^{(k)}$ is 2^{n^k-1}.*

Theorem 14. *Proper normal subgroups in $S_n \wr S_m$, where $n, m \geq 3$ with $n, m \neq 4$ are of the following types:*

(1) subgroups that act only on the second level are

$$\tilde{A}_m^{(1)},\ \widetilde{T_m},\ E \wr S_m, E \wr A_m,$$

(2) subgroups that act on both levels are $A_n \wr\wr \widetilde{A}_m^{(1)},\ S_n \wr\wr \widetilde{B}_m^{(1)},\ A_n \wr S_m,$
wherein the subgroup $S_n \wr\wr \widetilde{A_m} \simeq S_n \leftthreetimes (\underbrace{S_m \boxtimes S_m \boxtimes S_m \ldots \boxtimes S_m}_{n})$ *endowed with the subdirect product satisfying to condition* (2).

The group $\tilde{B}_n^{(1)}$ is isomorphic copy of $\tilde{A}_n^{(1)}$ which is realized by another embedding in $AutX^{[2]}$. The lattice of invariant subgroups for $S_n \wr S_n$, $n \equiv 0\,(\mathrm{mod}2)$ is presented on Fig. 1.

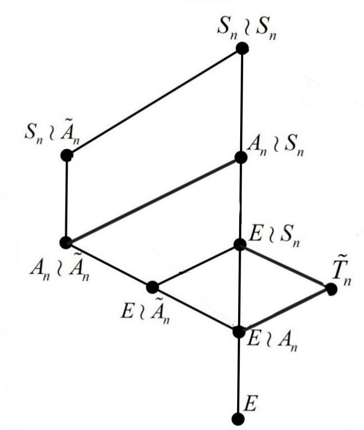

FIGURE 14.1. Lattice of invariant subgroups $S_n \wr S_n$ for the case $n \equiv 0\,(\mathrm{mod}2)$

Lemma 15. *Any 2 normal subgroups* $N_l, N_j \lhd \underset{i=1}{\overset{k}{\wr}} S_{n_i}, n_i \geqslant 3$ *are mutually commutative* $N_l N_j = N_j N_l$.

A group N of $AutX^{[k]}$ is said to be a group of depth $d = d\,(N)$ if N contain trivial permutations on levels $1, ..., d-1$, and first non-trivial permutation on level number $d \leq k$.

A group N of $AutX^{[k]}$ is said to be a group of height $h(N)$, where k is multiplicity of wreath product, if the difference $h = k - d\,(N)$, where $d\,(N)$ is depth of N. The set of normal subgroups of height h in $AutX^{[k]}$ is denoted by $N\,(h,k)$. Let us denote the number of normal subgroups of height h in $AutX^{[k]}$ as $n\,(h,k)$. We denote the i-th normal subgroup of height h in $AutX^{[k]}$ as $N_i\,(h,k)$. According to Theorem 14 $N(2,2) = \{N_i(2,2) : 1 \leq i \leq 5\}$.

Theorem 16. *The full list of normal subgroups of* $W = S_n \wr S_n \wr S_n \simeq AutX^{[3]}$ *consists of 50 normal subgroups.*

1 *subgroups of height 2 on base of set* $N(2,2)$ *takes form* $E \wr N_i(2,2)$*:* $E \wr A_n \wr\wr \widetilde{A}_n^{(1)}, E \wr A_n \wr S_n$ $E \wr S_n \wr\wr \widetilde{A}_n^{(1)}$, $E \wr S_n \wr\wr \widetilde{A}_n^{(1)}$ $E \wr S_n \wr\wr \widetilde{B_n^{(1)}}$, $E \wr S_n \wr S_n$. *There are 5 new subgroups.*

2 ***subgroups of height 2 in*** $AutX^{[3]}$ *that based on new subgroups of* X^3*:* $\tilde{A}_{n^2}$ *or* $\tilde{B}_{n^2}$*:* $E \wr S_n \wr\wr \tilde{A}_{n^2}$, $E \wr A_n \wr\wr \tilde{B}_{n^2}$, $E \wr A_n \wr\wr \tilde{A}_{n^2}$, $E \wr A_n \wr\wr \tilde{B}_{n^2}$, $E \wr \tilde{A}_n^{(2)} \wr\wr \tilde{A}_{n^2}^{(2)}$, *and subclass with subgroup*

H_i *on* X^1 *such that* $H_i \in \{\tilde{A}_n^{(1)}, \tilde{B}_n^{(1)}, \tilde{T}_n^{(1)}\}$ *so subgroups takes form* $E \wr\wr H_i \wr\wr \tilde{A}_{n^2}$, $E \wr\wr H_i \wr\wr \tilde{B}_{n^2}$. *Therefore this class has 10 new subgroups.*

3 subgroups of $N(2,3)$ *having a level subgroups on* $X^2 \subset X^{[3]}$ *such that from last level of* $N(2,2)$ *and one of them on* X^3 *There are 3 subgroup such level subgroups* $H_i \in \{\tilde{A}_n, \tilde{B}_n, S_n\}$. *Thus, there are 9 subgroups of form:* $E \wr \widetilde{A}_n \wr H_i, E \wr \widetilde{T}_n \wr H_i$, $E \wr \widetilde{B}_n \wr H_i$.

Thus, the total number of normal subgroups in of height 2 is 24.

4 subgroups of height 1 based on normal subgroups of type $N(1,2)$: $\prod_{i=1}^{9} A_n$, $\widetilde{T}_n$, $\widetilde{A}_n$, $\widetilde{B}_n$, $\prod_{i=1}^{9} S_n$. *And new subgroups of type* $N(1,3)$ $\widetilde{A}_{n^2}$, $\widetilde{B}_{n^2}$, $\widetilde{T_n}^{(2)}$. *Hence, here are 8 new subgroups.*

5 subgroups of height 3 admit on first level S_n *or* A_n, *on second one of* $\{\tilde{A}_n, \tilde{B}_n, S_n^3\}$, *on third* $\{\tilde{A}_{n^2}, \tilde{B}_{n^2}, S_n^9\}$. *Thus, there 18 normal subgroups in* $N(3,3)$.

Remark 17. Note that $E \wr\wr \widetilde{A}_n^{(1)} \simeq \widetilde{A}_n^{(2)}$ contains in the family $E \wr N_i(S_n \wr S_n)$.

We denote by $Aut_f X^*$ the group of all finite automorphism of spherically homogeneous rooted tree.

Theorem 18. *Let* $H \lhd Aut_f X^*$ *having depth* k, *then* H *contains* k*-th level subgroup* P *having all even vertex permutations* $p_{ki} \in A_n$ *on* X^k *and trivial permutations in vertices of rest of levels.*

Furthermore P *is normal in* $Aut_f X^*$ *provided* k *is last active level of* $Aut_f X^*$.

Theorem 19. *The order of normal subgroup* $\widetilde{A}_{n^j}^{(1,k)}$ *is* $(\frac{1}{2})^k \cdot (n!)^{(\frac{n^{(k+1)}-1}{n-1})}$ *and the order of the quotient* $\mathop{\wr}\limits_{i=1}^{k} S_{n_i} \Big/ \widetilde{A}_{n^j}^{(1,k)}$ *is* 2^k. *The order of generalized alternating group of* k*-th level* $\widetilde{A}_{n^k}^{(k)}$ *is* 2^{n^k-1}.

To study the parity of elements at all levels, we factorize by the *maximal normal subgroup* $\tilde{A}_{n^i}^{1,k}$ that contains the *generalized alternating group* of permutations at each level.

Lemma 20. *The following homomorphism* $W_n/\tilde{A}_{n^k}^k \cong W_{n-1} \wr C_2$ *holds.*

Theorem 21. *The quotient* $\mathop{\wr}\limits_{i=1}^{k} S_{n_i}$ *by* $\widetilde{A}_{n^j}^{(1,k)}$ *is the following group* $\prod\limits_{i=1}^{k} \mathbb{Z}_2$.

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Categorical Version of Shilov's Theorem on Closed Equivalence Relations

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Let X be a compact Hausdorff topological space, $A = C(X)$, and let B be a closed self-adjoint subalgebra of the algebra A. Then the classic Shilov's theorem says that equivalence relation R_B on the space X, defined by the formula:

$$R_B = \{(x, y) \in X \times X : \forall b \in B,\ b(x) = b(y)\}$$

is a closed equivalence relation, and B is the algebra of functions invariant under R_B.

This work explores the relationship between the category of closed equivalence relations on compact topological spaces and the category of pairs of commutative C^*-algebras. It is shown that these categories are equivalent.

To describe this equivalence, the functors C and Σ are constructed. The functor C assigns to an object (X, R) in the category of closed equivalence relations the pair $(C(X), B_R)$, where $C(X)$ is the algebra of continuous functions on X and B_R is the algebra of invariant functions with respect to R. The functor Σ assigns to an object (A, B) in the category of pairs of commutative C^*-algebras the spectrum of this pair, that is spectrum $\Sigma(A)$ of the algebra A and a natural equivalence relation on $\Sigma(A)$.

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On three-dimensional equidistant pseudo-Riemannian spaces

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A pseudo-Riemannian space V_n with metric tensor g_{ij} is called equidistant if there exists in it a vector field $\phi_i \not\equiv 0$ (also called an equidistance vector) that satisfies the equation

$$\phi_{i,j} = \tau g_{ij}. \tag{1}$$

where τ is some invariant, and the comma "," is the sign of the covariant derivative in V_n. When $\tau \neq 0$ this is the equidistant space of the basic case, and when $\tau = 0$ it is the special case [1].

The integrability conditions of the basic equations (1) have the form

$$\phi_\alpha R^{\alpha}_{\cdot\, ijk} = \tau_{,k} g_{ij} - \tau_{,j} g_{ik}. \tag{2}$$

In equidistant spaces, the equidistant vector is proportional to the tensor $\tau_{,i}$. Since $\phi_i \not\equiv 0$, there exists a vector ξ_i such that the convolution $\phi_\alpha \xi^\alpha = 1$ and, then, from the integrability conditions (2) it is not difficult to obtain that

$$\tau_{,i} = B\phi_i, \;\; B \stackrel{def}{=} \tau_{,\alpha}\xi^\alpha. \tag{3}$$

From these same conditions, multiplying by g^{ij} and folding over the indices i, j, we obtain

$$\tau_{,i} = \frac{1}{(n-1)} \phi_\alpha R^{\alpha}_{\cdot\, i} \tag{4}$$

For $n = 3$ the curvature tensor in pseudo-Riemannian spaces has the following form [2]:

$$R_{ijkl} = R_{il} g_{jk} - R_{ik} g_{jl} + R_{jk} g_{il} - R_{jl} g_{ik} - \frac{R}{2}(g_{il} g_{jk} - g_{ik} g_{jl}). \tag{5}$$

Next, let's consider (4) in (3) for $n = 3$

$$\phi_\alpha R^{\alpha}_{\cdot\, i} = 2B\phi_i \tag{6}$$

Taking into account equidistance and (6), from (5) we obtain the following identity

$$\phi_k (g_{jl}(\frac{R}{2} - B) - R_{jl}) - \phi_l (g_{jk}(\frac{R}{2} - B) - R_{jk}) = 0. \tag{7}$$

From here we obtain the following form for the Ricci tensor

$$R_{jl} = \phi_j \phi_l \Phi + g_{jl}(\frac{R}{2} - B), \;\; \Phi \stackrel{def}{=} \xi_\alpha \xi_\beta R^{\alpha\beta} - \xi_\alpha \xi^\alpha (\frac{R}{2} - B) \tag{8}$$

and then we substitute this into (5) and obtain the form for the curvature tensor

$$R_{ijkl} = \Phi(\phi_i \phi_l g_{jk} - \phi_i \phi_k g_{jl} + \phi_j \phi_k g_{il} - \phi_j \phi_l g_{ik}) + (\frac{R}{2} - 2B)(g_{jk} g_{il} - g_{jl} g_{ik}). \tag{9}$$

Thus, a necessary condition is obtained for a pseudo-Riemannian space to be three-dimensional and equidistant. Formulas (8) and (9) allow us to more effectively investigate objects of these spaces, mappings, etc.

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On the asymptotic behavior at infinity of solutions of the Beltrami equation with two characteristics

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Let D be a domain in the complex plane $\mathbb{C}$, i.e., a connected and open subset of $\mathbb{C}$, and let μ and $\nu\colon D \to \mathbb{C}$ be a measurable functions with $|\mu(z)| + |\nu(z)| < 1$ a.e. (almost everywhere) in D. We study the *Beltrami equation with two characteristics*

$$f_{\overline{z}} \;=\; \mu(z) f_z + \nu(z)\overline{f_z} \quad \text{a.e.} \quad \text{in} \quad D, \tag{1}$$

where $f_{\overline{z}} = (f_x + i f_y)/2$, $f_z = (f_x - i f_y)/2$, $z = x + iy$, f_x and f_y are the partial derivatives of f by x and y, respectively. The functions μ and ν are called the *complex coefficients* and

$$K_{\mu,\nu}(z) := \frac{1 + |\mu(z)| + |\nu(z)|}{1 - |\mu(z)| - |\nu(z)|}$$

the *dilatation quotient* for the equation (1).

Picking $\nu(z) \equiv 0$ in (1), we arrive at the standard *Beltrami equation* of the form

$$f_{\overline{z}} \;=\; \mu(z) f_z. \tag{2}$$

For the equation (2) we set

$$K_\mu(z) \;=\; \frac{1 + |\mu(z)|}{1 - |\mu(z)|}.$$

Picking $\mu(z) \equiv 0$ in (1), we arrive at the *Beltrami equation of the second type*

$$f_{\overline{z}} \;=\; \nu(z)\overline{f_z}. \tag{3}$$

For the equation (3) we set

$$K_\nu(z) \;=\; \frac{1 + |\nu(z)|}{1 - |\nu(z)|}.$$

Let $z_0 \in \mathbb{C}$ and $r > 0$. We put $B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$.

We say that a function $\varphi\colon \mathbb{C} \to \mathbb{R}$ has a *global finite mean value* at the point $z_0 \in \mathbb{C}$, abbr. $\varphi \in GFMV(z_0)$, if

$$\limsup_{R\to\infty} \frac{1}{\pi R^2} \int\limits_{B(z_0,R)} |\varphi(z)|\, dxdy < \infty.$$

For homeomorphism $f:\mathbb{C}\to\mathbb{C}$ we put

$$L_f(z_0,r)=\max_{|z-z_0|=r}|f(z)-f(z_0)|,\quad l_f(z_0,r)=\min_{|z-z_0|=r}|f(z)-f(z_0)|.$$

Theorem 1. *Let μ and $\nu\colon\mathbb{C}\to\mathbb{C}$ be a measurable functions with $|\mu(z)|+|\nu(z)|<1$ a.e. and $f\colon\mathbb{C}\to\mathbb{C}$ be a homeomorphic $W^{1,1}_{\rm loc}$ solution of the Beltrami equation (1), $z_0\in\mathbb{C}$. Assume that $K_{\mu,\nu}\in GFMV(\mathbb{C})$ and*

$$k_\infty=k_\infty(z_0)=\sup_{R\in(e,+\infty)}\frac{1}{\pi R^2}\int\limits_{B(z_0,R)}K_{\mu,\nu}(z)\,dxdy,$$

then

$$\liminf_{R\to\infty}\frac{L_f(z_0,R)}{R^p}\geqslant c\,l_f(z_0,e),$$

where $p=\frac{2}{e^2k_\infty}$ and $c=e^{-\frac{4}{e^2k_\infty}}$.

Picking $\nu(z)\equiv 0$ in Theorem 1, we arrive at the following statement.

Theorem 2. *Let $\mu\colon\mathbb{C}\to\mathbb{C}$ be a measurable function with $|\mu(z)|<1$ a.e. and $f\colon\mathbb{C}\to\mathbb{C}$ be a homeomorphic $W^{1,1}_{\rm loc}$ solution of the Beltrami equation (2), $z_0\in\mathbb{C}$. Assume that $K_\mu\in GFMV(\mathbb{C})$ and*

$$k_\infty=\sup_{R\in(e,+\infty)}\frac{1}{\pi R^2}\int\limits_{B(z_0,R)}K_{\mu}(z)\,dxdy,$$

then

$$\liminf_{R\to\infty}\frac{L_f(z_0,R)}{R^p}\geqslant c\,l_f(z_0,e),$$

where $p=\frac{2}{e^2k_\infty}$ and $c=e^{-\frac{4}{e^2k_\infty}}$.

Letting $\mu(z)\equiv 0$ in Theorem 1, we derive the following statement.

Theorem 3. *Let $\nu\colon\mathbb{C}\to\mathbb{C}$ be a measurable function with $|\nu(z)|<1$ a.e. and $f\colon\mathbb{C}\to\mathbb{C}$ be a homeomorphic $W^{1,1}_{\rm loc}$ solution of the Beltrami equation (3), $z_0\in\mathbb{C}$. Assume that $K_\nu\in GFMV(\mathbb{C})$ and*

$$k_\infty=\sup_{R\in(e,+\infty)}\frac{1}{\pi R^2}\int\limits_{B(z_0,R)}K_{\nu}(z)\,dxdy,$$

then

$$\liminf_{R\to\infty}\frac{L_f(z_0,R)}{R^p}\geqslant c\,l_f(z_0,e),$$

where $p=\frac{2}{e^2k_\infty}$ and $c=e^{-\frac{4}{e^2k_\infty}}$.

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Space-like minimal surface in Minkowski space and their Grassman image

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There is a coordinate system in Minkowski space ${}^1R_4$ for which the metric of the space has the form $ds^2 = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$. Let the equation $\bar{r} = \bar{r}(u^1, u^2)$ defines a two-dimensional space-like surface F^2, the vectors $\bar{\xi}_1, \bar{\xi}_2$ are respectively its time-like and space-like normal vectors, and g_{ij}, L^k_{ij} are respectively the coefficients of the first and second quadratic forms. The number $H^k = \frac{1}{2} g^{ij} L^k_{ij}$ is called the mean curvature of the surface for the direction of the normal $\bar{\xi}_k$, and the vector $H = H^1 \bar{\xi}_1 + H^2 \bar{\xi}_2$ is the mean curvature vector. Space-like surfaces of Minkowski space with zero mean curvature vector will be called minimal surfaces, as in Euclidean space.

We plan to apply the concept of the indicatrix of the normal curvature of a surface to the study of its differential geometry, in particular, the question of the existence of such surfaces with some additional conditions on their Grassman image. The question of the existence of a time-like minimal surface with a constant curvature of its Grassman image was solved in work [3].

At point x on a space-like surface F^2 each direction $\bar{\tau} \in T_x F^2$ corresponds to a normal curvature vector $k(\bar{\tau}) = (Pr_{\bar{\xi}_1}, \bar{r}_{ss})\bar{\xi}_1 + (Pr_{\bar{\xi}_2}, \bar{r}_{ss})\bar{\xi}_2 = -(\bar{r}_{ss}, \bar{\xi}_1)\bar{\xi}_1 + (\bar{r}_{ss}, \bar{\xi}_2)\bar{\xi}_2 = -\frac{II^1}{ds^2}\bar{\xi}_1 + \frac{II^2}{ds^2}\bar{\xi}_2$, where $\bar{r}_{ss}$ is the curvature vector of the curve on the surface F^2 at the point x, which has the direction $\bar{\tau}$, and the scalar projections are defined by the formulas $Pr_{\bar{\xi}_i}, \bar{r}_{ss} = sign(\bar{\xi}_i^2)\frac{(\bar{r}_{ss}, \bar{\xi}_i)}{\sqrt{|\bar{\xi}_i^2|}}$. When direction $\bar{\tau}$ rotates in the tangent plane $\bar{\tau} \in T_x F^2$, the end $P(-\frac{II^1}{ds^2}; \frac{II^2}{ds^2})$ of the vector $\bar{r}_{ss}$ will form a curve, which we will call the indicatrix of normal curvature by analogy with Euclidean space [1].

Let us move on to such parameterization u^1, u^2 of the surface, for which the metric tensor has the form $g_{ij} = \delta_{ij}$. Next, we select a point (α, β), $\alpha = -\frac{L^1_{11} + L^1_{22}}{2}$, $\beta = -\frac{L^2_{11} + L^2_{22}}{2}$ in the plane N_x as the origin of the coordinate system. The geometric meaning of this transfer is to move to a raper with the origin in the center of the normal curvature indicatrix. Next, we choose normals $\bar{\xi}_1, \bar{\xi}_2$ parallel to the axes of the indicatrix, introduce the notations $\frac{L^1_{11} - L^1_{22}}{2} = a$, $L^2_{12} = b$, and obtain expressions for the coefficients of the second quadratic forms in the form $L^1_{11} = -(\alpha - a)$, $L^1_{12} = 0$, $L^1_{22} = -(\alpha + a)$, $L^2_{11} = \beta$, $L^2_{12} = b$, $L^2_{22} = \beta$.

The Grassmann image of two-dimensional surfaces is an important geometric characteristic of them. In [2] it is shown that the nondegenerate Grassmann image Γ^2 of a surface of Minkowski space is a two-dimensional surface $\bar{p} = \bar{p}(u^1, u^2)$, which belongs to the four-dimensional Grassmann submanifold $PG(2, 4)$ of the six-dimensional pseudo-Euclidean space ${}^3R_6$ of index 3. Tangent vectors to Γ^2 can be written in the form $\bar{p}_{u_i} = -L^1_{ik} g^{kl} [\bar{r}_l, \bar{\xi}_2] - L^2_{ik} g^{kl} [\bar{\xi}_1, \bar{r}_l], l = 1, 2$.

From the condition $g^{ij} L^k_{ij} = 0$ of minimality of the surface it follows that $\alpha = \beta = 0$. The metric of the Grassmann image of the minimal space-like surface of the space ${}^1R_4$ in the parameters of the normal curvature indicatrix has the form $ds^2 = (a^2 - b^2)^2 g_{11} g_{22} du^1 du^2$, and therefore the Grassmann image of the surface is also a space-like surface. The formula for the sectional curvature of the Grassmann image has the form $\overline{K} = -1 + \frac{4a^2 b^2}{(a^2 - b^2)^2}$ and therefore it can take on values from the interval $(-1; +\infty)$.

To solve the problem of the existence of space-like minimal surfaces with a non-degenerate Grassmann image of constant curvature $\overline{K}$, it is necessary to prove that under this condition the system of Gauss-Kodazzi-Ricci equations

$$\begin{cases} R_{1212} = (a^2 - b^2)^2 g_{11} g_{22}, \\ (ag_{11})'_{u^2} = b\sqrt{g_{11}g_{22}}\mu_{12/1}, \\ (ag_{22})'_{u^1} = -b\sqrt{g_{11}g_{22}}\mu_{12/2}, \\ (b\sqrt{g_{11}g_{22}})'_{u^1} = -ag_{11}\mu_{12/2}, \\ (b\sqrt{g_{11}g_{22}})'_{u^2} = ag_{11}\mu_{12/1}, \\ (\mu_{12/1})'_{u^2} - (\mu_{12/2})'_{u^1} = -2ab\sqrt{g_{11}g_{22}}, \end{cases} \tag{1}$$

is compatible. Here $\mu_{12/i}$ are the torsion coefficients.

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Fejer's method in extremal problems of geometric complex analysis

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For the class of typically real in the unit disc of the complex plane polynomials, the results of W. Rogosinski and G. Szegö [1] implies the sharp estimates for the second coefficient, however the problem of finding the extremizers still open.

Within algebraic framework, we construct explicit polynomials which attain these bounds and prove their uniqueness. The proof uses the Fejér-Riesz representation of nonnegative trigonometric polynomials, a 7-band Toeplitz matrix of arbitrary finite dimension, and Chebyshev polynomials of the second kind and their derivatives.

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On convergence of homeomorphisms with inverse Poletsky inequality

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Below $dm(x)$ denotes the element of the Lebesgue measure in $\mathbb{R}^n$. Everywhere further the boundary ∂A of the set A and the closure $\overline{A}$ should be understood in the sense of the extended Euclidean space $\overline{\mathbb{R}^n}$. Recall that, a Borel function $\rho : \mathbb{R}^n \to [0,\infty]$ is called *admissible* for the family Γ of paths γ in $\mathbb{R}^n$, if the relation

$$\int\limits_{\gamma} \rho(x)\,|dx| \geqslant 1$$

holds for all (locally rectifiable) paths $\gamma \in \Gamma$. In this case, we write: $\rho \in \operatorname{adm}\Gamma$. The *modulus* of Γ is defined by the equality

$$M(\Gamma) = \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^n(x)\,dm(x)\,.$$

Let $y_0 \in \mathbb{R}^n$, $0 < r_1 < r_2 < \infty$ and

$$A = A(y_0, r_1, r_2) = \{y \in \mathbb{R}^n : r_1 < |y - y_0| < r_2\}\,.$$

Given $x_0 \in \mathbb{R}^n$, we put $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$, $\mathbb{B}^n = B(0,1)$, $S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}$. A mapping $f : D \to \mathbb{R}^n$ is called *discrete* if the pre-image $\{f^{-1}(y)\}$ of any point $y \in \mathbb{R}^n$ consists of isolated points, and *open* if the image of any open set $U \subset D$ is an open set in $\mathbb{R}^n$. Given sets E, $F \subset \overline{\mathbb{R}^n}$ and a domain $D \subset \mathbb{R}^n$ we denote by $\Gamma(E, F, D)$ the family of all paths $\gamma : [a, b] \to \overline{\mathbb{R}^n}$ such that $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for $t \in (a, b)$. Given a mapping $f : D \to \mathbb{R}^n$, a point $y_0 \in \overline{f(D)} \setminus \{\infty\}$, and $0 < r_1 < r_2 < r_0 = \sup\limits_{y \in f(D)} |y - y_0|$, we denote by $\Gamma_f(y_0, r_1, r_2)$ a family of all paths γ in D such that $f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$. Let $Q : \mathbb{R}^n \to [0,\infty]$ be a Lebesgue measurable function. We say that f *satisfies the inverse Poletsky inequality at a point* $y_0 \in \overline{f(D)} \setminus \{\infty\}$ if the relation

$$M(\Gamma_f(y_0, r_1, r_2)) \leqslant \int\limits_{A(y_0,r_1,r_2)\cap f(D)} Q(y)\cdot \eta^n(|y - y_0|)\,dm(y) \tag{1}$$

holds for any Lebesgue measurable function $\eta : (r_1, r_2) \to [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\,dr \geqslant 1\,. \tag{2}$$

The relations (1) are proved for different classes of mappings, see e.g. [1].

Set $q_{y_0}(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{S(y_0,r)} Q(y)\,d\mathcal{H}^{n-1}(y)$, where ω_{n-1} denotes the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. We say that a function $\varphi : D \to \mathbb{R}$ has a *finite mean oscillation* at a point $x_0 \in D$, write

$\varphi \in FMO(x_0)$, if $\limsup\limits_{\varepsilon\to 0} \frac{1}{\Omega_n\varepsilon^n} \int\limits_{B(x_0,\,\varepsilon)} |\varphi(x)-\overline{\varphi}_\varepsilon|\,dm(x) < \infty$, where $\overline{\varphi}_\varepsilon = \frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(x_0,\,\varepsilon)} \varphi(x)\,dm(x)$ and Ω_n is the volume of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$. We also say that a function $\varphi : D\to\mathbb{R}$ has a finite mean oscillation at $A\subset \overline{D}$, write $\varphi\in FMO(A)$, if φ has a finite mean oscillation at any point $x_0\in A$. Let h be a chordal metric in $\overline{\mathbb{R}^n}$,

$$h(x,\infty)=\frac{1}{\sqrt{1+{|x|}^2}}\,,\quad h(x,y)=\frac{|x-y|}{\sqrt{1+{|x|}^2}\sqrt{1+{|y|}^2}}\,,\qquad x\ne\infty\ne y\,,$$

and let $h(E):=\sup\limits_{x,y\in E} h(x,y)$ be a chordal diameter of a set $E\subset\overline{\mathbb{R}^n}$ (see, e.g., [2, Definition 12.1]).

Theorem 1. *Let D be a domain in $\mathbb{R}^n$, $n\geqslant 2$, and let $f_j : D\to\mathbb{R}^n$, $j=1,2,\ldots,$ be a sequence of homeomorphisms that converges to some mapping $f : D\to\overline{\mathbb{R}^n}$ locally uniformly in D by the metric h, and satisfy the relations (1)–(2) in each point $y_0\in\overline{\mathbb{R}^n}$. Assume that one of two conditions holds:*

1) $Q\in FMO(\overline{\mathbb{R}^n})$, or

2) for any $y_0\in\overline{\mathbb{R}^n}$ there exist $\varepsilon_1(y_0)>0$ and $\delta(y_0)>0$ such that

$$\int\limits_{\varepsilon}^{\delta(y_0)}\frac{dt}{tq_{y_0}^{\frac{1}{n-1}}(t)}<\infty\qquad\forall\,\varepsilon\in(0,\varepsilon_1(y_0)),\qquad \int\limits_{0}^{\delta(y_0)}\frac{dt}{tq_{y_0}^{\frac{1}{n-1}}(t)}=\infty\,. \tag{3}$$

Then f is either a homeomorphism $f : D\to\mathbb{R}^n$, or a constant $c\in\overline{\mathbb{R}^n}$.

Here the conditions mentioned above for $y_0=\infty$ must be understood as conditions for the function $\widetilde{Q}(y):=Q(y/|y|^2)$ at the origin. We should note that the second condition in (3) is not only a sufficient but also a necessary condition in Theorem 1. The following conclusion holds.

Theorem 2. *Let $Q:\mathbb{R}^n\to[0,\infty]$ be locally integrable function such that*

$$\int\limits_{0}^{\delta(y_0)}\frac{dt}{tq_{y_0}^{\frac{1}{n-1}}(t)}<\infty$$

for some $y_0\in\mathbb{R}^n$ and $\delta(y_0)>0$. Then there exists a sequence of homeomorphisms $f_j : D\to\mathbb{R}^n$, $j=1,2,\ldots,$ satisfying the relations (1)–(2) at y_0 which converges to some mapping $f : D\to\overline{\mathbb{R}^n}$ locally uniformly in D by the metric h, which is neither a homeomorphism nor a constant.

The results mentioned above are published in [3].

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Are Totally Convex Surfaces Area Minimizing? (A note on definitions and counterexamples)

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We investigate the claim that a compact surface Σ with boundary $\partial\Sigma$, embedded in a manifold M, is area minimizing among all surfaces with the same boundary if it satisfies a "total convexity" property. This problem connects methods from geometric analysis and algebraic geometry. We clarify two main interpretations of total convexity: intrinsic (geodesic) and extrinsic (ambient). Under the intrinsic definition, we demonstrate the claim is false using the counterexample of a spherical cap. For the extrinsic definition which implies Σ is a convex domain in a totally geodesic submanifold $P \subset M$ the claim holds in Euclidean space (proven via calibration using differential forms) and is generally true for hypersurfaces in Riemannian manifolds where stability analysis is more tractable. However, it can fail in higher codimension due to more complex stability criteria. We provide justifications, citing known counterexamples from minimal surface theory, particularly those in normed spaces, which underscore the subtleties. Connections to rigidity problems are briefly explored, highlighting how specific geometric and algebraic structures might enforce uniqueness.

Anabelian geometry in arithmetic topology

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This talk is based on a joint work with Nadav Gropper and Yi Wang [GUW25].

The analogy between knots and primes, or 3-manifolds and the ring of integers of number fields, has been systematically developed by Mazur [Maz64, Maz12], Kapranov [Kap95], Reznikov [Rez97, Rez00], Morishita [Mor02, Mor12, Mor24], Kim [Kim20], and others. In their spirit of *arithmetic topology*, we have formulated in [Nii14, NU19] an analogue of Artin–Takagi–Chevalley's *idelic class field theory* that sums up all local theories to describe all abelian branched covers of a 3-manifold M endowed with a certain infinite link $\mathcal{K}$. Successive studies are [Mih19, NU23, Tas25b, Tas25a]. In addition, analogues of the set of all primes have been studied in [Maz12, McM13, Uek20, Uek21a, Uek21b].

Extending this context, we may discuss an analogue of so-called *anabelian geometry*, whose initial fundamental result is the classical *Neukirch–Uchida theorem* stated as follows.

Theorem 1 (Neukirch [Neu69b, Neu69a], Uchida [Uch76], see also [NSW08, Theorem 12.2.1])**.** *Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. Let E, F be number fields, that is, finite extensions of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$. If there is an isomrphism $\varphi : \mathrm{Gal}(\overline{\mathbb{Q}}/E) \overset{\cong}{\to} \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ of topological groups, then there uniquely exists a natural isomorphism $E \overset{\cong}{\to} F$, that is, there is a unique $\sigma \in \mathrm{Aut}\,\overline{\mathbb{Q}}$ such that $F = \sigma(E)$ and σ induces φ.*

In the proof, the Hilbert ramification theory for infinite Galois extensions, the Poiteau–Tate duality, and the Chebotarev density theorem play key roles. One of the main steps is to prove the following.

Theorem 2 ([NSW08, Theorem 12.2.5])**.** *Let $F/\mathbb{Q}$ be a finite Galois extension and $E/\mathbb{Q}$ a finite extension. If all primes $p \in \mathbb{Q}$ with a prime factor of degree 1 in $E/\mathbb{Q}$ completely decompose in $F/\mathbb{Q}$, then $F \subset E$.*

Now let M be an oriented connected closed 3-manifold with a base point b_M and let $\mathcal{K} = \bigcup_{i \in \mathbb{Z}_{\geq 0}} K_i$ be an infinite link consisting of countably many tame components. Let $\mathrm{Cov}(M, \mathcal{K})$ denote the set of all branched covers branched along finite sublinks of $\mathcal{K}$. We define *the absolute Galois group* of $(M, \mathcal{K})$ by $\mathrm{Gal}(M, \mathcal{K}) = \varprojlim_{h \in \mathrm{Cov}(M,\mathcal{K})} \mathrm{Gal}\, h = \varprojlim_{L \subset \mathcal{K}} \widehat{\pi}_1(M - L)$, where $\widehat{\pi}_1$ denotes the profinite completion of π_1. Then, we may formulate *the Hilbert ramification theory* for pro-covers [GUW25]. Suppose in addition that $\mathcal{K}$ obeys *the Chebotarev law*. Then it turns out that for any $h \in \mathrm{Cov}(M, \mathcal{K})$, the inverse image $h^{-1}(\mathcal{K})$ is again Chebotarev [GUW25]. An analogue of Theorem 1 may be stated as follows.

Theorem 3 ([GUW25])**.** *Let the setting be as above. Let G_1, G_2 be open subgroups of* $\mathrm{Gal}(M, \mathcal{K})$ *and let $h_1, h_2 \in \mathrm{Cov}(M, \mathcal{K})$ denote the corresponding branched covers. If there is an isomorphism $\varphi : G_1 \overset{\cong}{\to} G_2$ of topological groups, then there uniquely exists a natural isomorphism $h_1 \cong h_2$ of branched covers, that is, there is a unique $\sigma \in \mathrm{Gal}(M, \mathcal{K})$ such that $h_2 \circ \sigma = h_1$ and σ induces φ.*

An analogue of the key step is as follows.

Theorem 4 ([GUW25])**.** *Let $h_1, h_2 \in \mathrm{Cov}(M, \mathcal{K})$ and suppose that h_1 is Galois. If all knots $K \subset \mathcal{K}$ whose inverse image $h_2^{-1}(K)$ has a component of covering degree 1 in h_2 completely decompose in h_1, then h_1 is a subcover of h_2.*

Once the theorem's statement comes into view, in the context of research aiming to systematize analogies, numerous problems to be addressed in the future become apparent. In the topology side, the classical Mostow rigidity assures that hyperbolic manifolds are determined by their fundamental groups. In addition, in recent days, profinite rigidity has been of great interest [Rei18, BJZR23]. But we believe that rigidity for such a large group $\mathrm{Gal}(M, \mathcal{K})$ is a new viewpoint and would be of interest, even away from the context of the analogy.

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On geometries of Kac-Moody groups and Symbolic Computations

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Let K be a commutative ring with the unity. Jordan-Gauss graph over K is an incidence structures with partition sets P (points) and L (lines) isomorphic to affine spaces V_1 and V_2 over K such that the incidence relation is given by special quadratic equations over the commutative ring K with unity such that the neighbour of each vertex is defined by the system of linear equation given in its row-echelon form.

We assume that V_i, $i = 1, 2$ are finite dimensional spaces of kind K^n or infinite dimensional affine spaces formed tu tuples with finite support.

Let Γ be an incidence system with partition sets Γ_i, $i = 1, 2, \ldots, m$ and incidence relation I. We say that equivalence τ on Γ is Jordan-Gauss equivalence over K if each equivalence class is an affine space over this ring, each Γ_i is a union of these equivalence classes and the restriction of I

on the union of two such classes consisting of elements of different types is a Jordan-Gauss graph or empty relation.

Theorem 1. *Let F be a field, $G(F)$ be a Kac-Moody group and $\Gamma(G(F))$ be a Kac - Moody geometry of $G(F)$. Then there is Jordan-Gauss equivalence on $\Gamma(G(F))$ defined over F. Totality of equivalence classes of this relation is in one to one correspondence with elements of corresponding Weyl geometry.*

The theorem is the corollary of the results presented in [1], [2].

Let B^+ and B^- be Borel subgroups containing root subgroups corresponding positive and negative roots respectively. Let P_i, $i = 1, 2, \dots, m$ are standard maximal parabolic subgroups, i. e maximal subgroups of G containing B^+. The geometry $\Gamma(G(F))$ is the disjoint union of $(G(F) : P_i)$ with the type function $t(gP_i) = i$ and incidence relation I : $\alpha I \beta$ if and only if $\alpha \cap \beta$ is not an empty set. Orbits of B^- form the classes of Jordan-Gauss equivalence relation.

Jordan-Gauss graph is the special case of linguistic graph given by the following way. We identify points with tuples of kind $(x) = (x_1, x_2, \dots, x_n, \dots)$ and lines with tuples $[y] = [y_1, y_2, \dots, y_n, \dots]$. Brackets and parenthesis are convenient to distinguished type of the vertex of the graph. Elements (x) and $[y]$ are incident $(x)I[y]$ if and only if the following relations hold.

$a_1 x_{s+1} - b_1 y_{r+1} = f_1(x_1, x_2, \dots, x_s, y_1, y_2, \dots, y_r)$,

$a_2 x_{s+2} - b_2 y_{r+2} = f_2(x_1, x_2, \dots, x_s, x_{s+1}, y_1, y_2, \dots, y_r, y_{r+1})$,

$\dots$

$a_m x_{s+m} - b_m y_{r+m} = f_m(x_1, x_2, \dots, x_s, x_{s+1}, \dots, x_{s+m-1}, y_1, y_2, \dots, y_r, y_{r+1}, \dots, y_{r+m-1})$

...

where a_j and b_j, $j = 1, 2, \dots, m$ are not zero divisors, and f_j are multivariate polynomials with coefficients from K.

Linguistic graph given by the written above equations is Jordan – Gauss graph if the map sending the pair $((x_1, x_2, \dots, x_n, \dots), (y_1, y_2, \dots, y_n, \dots))$ to $(f_1, f_2, \dots, f_m, \dots)$ is a bilinear one.

We use the interpretations of geometries $\Gamma(G(F))$ to define their analogues $\Gamma(G(K))$ where K is arbitrary commutative ring with unity. The walks on incidence structures $\Gamma(G(K)$ and natural colourings of their Jordan - Gauss graphs are used for the design of explicit constructions of groups supporting the following statement of Computational Algebraic Geometry.

Theorem 2. *Let K be commutative ring with unity. For each positive integer n, d, $d \geq 2$ and rational parameter $s \geq 0$ there is a subgroup H of affine Cremona semigroup of all endomorphisms of $K[x_1, x_2, \dots, x_n]$ such that maximal degree of representative of H is d and the densities of elements from H are of size $O(n^s)$.*

Recall that degree and density of endomorphism F of $K[x_1, x_2, \dots, x_n]$ is defined as maximal values of degrees and densities of standard forms of polynomials $F(x_i)$, $i = 1, 2, \dots, n$. For each commutative ring K with unity and selected positive integer d, $d > 1$ and rational parameter s, $d \geq s > 0$ we construct polynomial bijective map F of K^n onto K^n of degree d, density $O(n^s)$ with the computational accelerator T which is the piece of information such that its knowledge allows to compute the reimage of F in time $O(n^2)$.

During the talk some applications of these results to Computational Algebraic Geometry the Theory of Communications will be described (see [3]).

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Hidden symmetry-like structures of dispersionless Nizhnik equation

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In [3], L. P. Nizhnik studied the integration of multidimensional nonlinear equations using the inverse scattering method. Therein, the system

$$w_t = k_1 w_{xxx} + k_2 w_{yyy} + 3(v^1 w)_x + 3(v^2 w)_y, \quad v^1_y = k_1 w_x, \quad v^2_x = k_2 w_y \tag{1}$$

known now as the Nizhnik system was introduced. It has two parameters, k_1 and k_2 with $(k_1, k_2) \neq (0, 0)$, but the only thing that matters is whether the product $k_1 k_2$ is zero (the asymmetric case) or not (the symmetric case). Introducing potentials in the system (1) and/or taking limits with respect to a small scaling parameter, we can derive various related models,

- the (symmetric potential) Nizhnik equation $u_{txy} = u_{xxxxy} + u_{xyyyy} + 3(u_{xx}u_{xy})_x + 3(u_{yy}u_{xy})_y$,
- the asymmetric (potential) Nizhnik equation $u_{ty} = u_{xxxy} + 3(u_x u_y)_x$ (also called the Boiti–Leon–Manna–Pempinelli equation [1]),
- the (symmetric) dispersionless Nizhnik system $w_t = (v^1 w)_x + (v^2 w)_y$, $v^1_y = w_x$, $v^2_x = w_y$,
- the (symmetric potential) dispersionless Nizhnik equation
$$u_{txy} = (u_{xx}u_{xy})_x + (u_{xy}u_{yy})_y, \tag{2}$$
- the asymmetric dispersionless Nizhnik system $w_t = (v^1 w)_x$, $v^1_y = w_x$,
- the asymmetric (potential) dispersionless Nizhnik equation $u_{ty} = (u_x u_y)_x$.

The results from [2] on point symmetries of the equation (2) created a basis for a comprehensive classification of its Lie reductions to partial differential equations with two independent variables and to ordinary differential equations, which was carried out in [5]. The list of inequivalent one-dimensional subalgebras of the maximal Lie invariance pseudoalgebra $\mathfrak{g}$ of (2) presented in [5] includes, in particular, the family of subalgebras $\mathfrak{s}^{\rho}_{1.3} = \langle P^x(1) + P^y(\rho) \rangle$, where $\rho = \rho(t)$ is an arbitrary smooth function of t satisfying the inequalities $\rho(t) \neq 0$ for all t in its domain and $\rho \not\equiv 1$

on any open interval within that domain. The optimal ansatzes constructed with respect to these subalgebras reduce the equation (2) to partial differential equations in two independent variables that share the same form

$$w_{122} + w_{22}w_{222} = 0. \tag{3}$$

It is easy to see that the substitution $w_{22} = h$ maps the equation (3) to the inviscid Burgers equation

$$h_1 + hh_2 = 0. \tag{4}$$

The equation (3) is the most interesting and fruitful submodel of (2), in particular, in the sense of its relation to hidden symmetry-like objects of (2). In [4], we found all local symmetry-like objects associated with the equation (3), including generalized symmetries, cosymmetries, conservation-law characteristics and conservation laws, and most of them are hidden for (2). This represents the first comprehensive study of such objects for a submodel of a well-known system of differential equations. Complete descriptions even of particular kinds of such objects in nontrivial cases exist in the literature only for a minor part of these systems themselves, not to mention submodels. Moreover, a complete description of all the local symmetry-like objects of a model in a single paper is rather exceptional.

Standard techniques like recursion operators and the estimation of the dimension of the space of objects in question up to an arbitrary fixed order do not work for the equation (3). Even the best computer packages for finding local symmetry-like objects such as Jets and GeM for Maple are inefficient at computing such objects for this equation even at low orders, starting from order three. This can be explained by the fact that for local symmetry-like objects of any specific kind, the corresponding space of them for the equation (3) is of complicated structure. In particular, they are parameterized by functions of arbitrary finite number of arguments that are cumbersome differential expressions.

To illustrate the above claims, here we present only the description of generalized symmetries of (3).

Theorem 1. *A differential function $f\{w\}$ is the characteristic of a generalized symmetry of the equation (3) if and only if it is a linear combination of the differential functions*

$$w_{1,0}, \quad z_1w_{1,0} + w_{0,0}, \quad z_1^2w_{1,0} + z_1z_2w_{0,1} - z_1w_{0,0} - \frac{1}{6}z_2^3,$$
$$w_{0,1}, \quad 2z_1w_{0,1} - z_2^2, \quad z_2w_{0,1} - 3w_{0,0}, \quad \breve{g}, \quad z_2g, \quad \frac{w_{0,2}}{w_{1,2}\theta^2}(\breve{f}_{w_{0,2}} - \theta^{k+1}\breve{f}_{\theta^k}) + \breve{f},$$
$$\frac{(w_{0,2})^3}{2w_{1,2}}\frac{(\theta^1)^2}{\theta^2} + \frac{1}{6}z_1^2(w_{0,2})^3 + z_1w_{1,0}, \quad \frac{2}{3}\frac{(w_{0,2})^4}{w_{1,2}}\frac{(\theta^1)^2}{\theta^2} + \frac{1}{6}z_1^2(w_{0,2})^4 + z_2w_{1,0} - z_2^2\zeta^{10},$$
$$2\frac{(w_{0,2})^3}{w_{1,2}}\frac{(\theta^1)^2}{\theta^2}\theta^0 + \frac{2}{3}z_1^2(w_{0,2})^3\theta^0 + \frac{1}{6}z_1^3(w_{0,2})^4 + z_2w_{0,0} - z_2^2w_{0,1} - z_1z_2w_{1,0} + z_1z_2^2\zeta^{10}.$$

Here $w_{k,l} := \partial^{k+l}w/\partial z_1^k\partial z_2^l$, g and $\breve{g}$ are arbitrary functions of z_1 and a finite number of ζ^{ik}, the $\breve{f}$ is an arbitrary function of $w_{0,2}$ and a finite number of θ^k, $k, l \in \mathbb{N}_0$, $i = 1, 2$,

$$\zeta^{ik} := \mathrm{D}_1^kI^i, \quad \theta^k := \left(\frac{w_{0,2}}{w_{1,2}}\hat{\mathrm{D}}_2\right)^k(z_2 - w_{0,2}z_1),$$
$$I^1 := w_{1,1} + \frac{1}{2}(w_{0,2})^2, \quad I^2 := w_{2,0} - \frac{1}{3}(w_{0,2})^3 - z_2(w_{2,1} + w_{0,2}w_{1,2}) = w_{2,0} - \frac{1}{3}(w_{0,2})^3 - z_2\mathrm{D}_1I^1,$$

D_1 *and* D_2 *denote the operators of total derivatives with respect to the variables* z_1 *and* z_2*, and* $\hat{\mathrm{D}}_2$ *is the restriction of* D_2 *to solutions of* (3)*,*

$$\hat{\mathrm{D}}_2 = \partial_{z_2} + w_{0,1}\partial_{w_{0,0}} + \left(\zeta^{10} - \frac{1}{2}(w_{0,2})^2\right)\partial_{w_{1,0}} + w_{0,2}\partial_{w_{0,1}} - \hat{\mathrm{D}}_1^k\left(\frac{w_{1,2}}{w_{0,2}}\right)\partial_{w_{k,2}}.$$

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Motivic Hilbert zeta functions of curve singularities and related invariants

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Let C and p be a reduced singular curve over $\mathbb{C}$ and its singular point respectively. We refer to the germ of C at p as a curve singularity and denote it by (C, p). Let $K_0(\mathrm{Var}_{\mathbb{C}})$ be the Grothendieck ring of complex algebraic varieties.

For a reduced curve singularity (C, p), the *motivic Hilbert zeta function* with support at p is defined as

$$Z^{\mathrm{Hilb}}_{C,p}(t) := \sum_{l=0}^{\infty}[C_p^{[l]}]t^l \in 1 + tK_0(\mathrm{Var}_{\mathbb{C}})[[t]] \tag{1}$$

where $C_p^{[l]}$ consits of length l subschemes of C supported at p. It is known that $Z^{\mathrm{Hilb}}_{C,p}(t)$ is rational (see [1]). We refer to $C_p^{[l]}$ as *the punctual Hilbert scheme of degree* l for the given curve singularity (C, p). In this talk, we consider the following assumtion:

Assumption 1. For a given curve singularity (C, p), any punctual Hilbert scheme $C_p^{[l]}$ admits an affine cell decomposition.

Remark 2. It is known that any irreducible plane curve singularity with one Puiseux pair satisfies Assumption 1 (see [6]).

Let $\mathbb{L}$ denote the class of the affine line $\mathbb{A}^1$ in $K_0(\mathrm{Var}_{\mathbb{C}})$.

Lemma 3. *Let* (C, p) *be a reduced curve singularity that satisfies Assumption* 1*. Then the class* $[C_p^{[l]}]$ *in* $K_0(\mathrm{Var}_{\mathbb{C}})$ *is a polynomial in* $\mathbb{C}[\mathbb{L}]$*. Furthermore, the Euler number* $\chi(C_p^{[l]})$ *is equal to the number of affine cells of* $C^{[l]}$*.*

By Lemma 3, we see that the motivic Hilbert zeta function (1) is an element of $\mathbb{C}[\mathbb{L}][[t]]$ under Assumption 1. Therefore, instead of $Z^{\mathrm{Hilb}}_{C,p}(t)$, we use the notation $Z^{\mathrm{Hilb}}_{C,p}(t,\mathbb{L})$.

Theorem 4. *Let (C,p) be a reduced curve singularity. If Assumption 1 holds, then we have*

$$Z^{\mathrm{Hilb}}_{C,p}(q,1)=\sum_{l=0}^{\infty}\chi(C_p^{[l]})q^l$$

where $\chi(C_p^{[l]})$ is the Euler number of $C_p^{[l]}$.

Let Γ be a semigroup and let $\mathrm{Mod}(\Gamma)$ denote the set of all Γ-semimodules. For a Γ-semimodule Δ, we define its codimension by $\mathrm{codim}(\Delta):=\#(\Gamma\setminus\Delta)$. The *generating function $I(\Gamma;q)$ of Γ-semimodules* is defined to be

$$I(\Gamma;q):=\sum_{\Delta\in\mathrm{Mod}(\Gamma)}q^{\mathrm{codim}(\Delta)}.$$

Theorem 5. *For an irreducible curve singularity (C,p) with one Puiseux pair, the following realtion holds*:

$$Z^{\mathrm{Hilb}}_{C,p}(q,1)=I(\Gamma;q)$$

Using our results, we clarify the relations among Motivic Hilbert zeta functions and other invariants. Below we focus on reduced plane curve singularities. Let $P(L_{C,p})$ be the HOMFLY polynomial of the oriented link $L_{C,p}$ associated with (C,p). The following relation was conjectured by Oblomkov and Shende in [4] and was finally proved by Maulik in [3]:

$$\sum_{l=0}^{\infty}\chi(C_p^{[l]})q^{2l}=\left(\frac{q}{a}\right)^{\mu-1}P(L_{C,p})\Big|_{a=0} \tag{2}$$

On the other hand, Shende [5] also proved the relation

$$\sum_{l=0}^{\infty}\chi(C_p^{[l]})q^{l}=\sum_{l=0}^{\delta}q^{\delta-l}(1-q)^{2h-1}\mathrm{deg}_p\mathbb{V}_h \tag{3}$$

where δ is the delta invariant of (C,p) and $\mathbb{V}_h$'s are the severi strata of the miniversal deformation of (C,p).

Consequently, the following fact follows from Theorem 4 and 5, along with the relations (2) and (3).

Theorem 6. *Here notations remain the same as above. If (C,p) is an irreducible plane curve singularity with one Puiseux pair, then we have*

$$Z^{\mathrm{Hilb}}_{C,p}(q^2,1)=I(\Gamma;q^2)=\left(\frac{q}{a}\right)^{\mu-1}P(L_{C,p})\Big|_{a=0}, \tag{4}$$

$$Z^{\mathrm{Hilb}}_{C,p}(q,1)=I(\Gamma;q)=\sum_{l=0}^{\delta}q^{\delta-l}(1-q)^{2h-1}\mathrm{deg}_p\mathbb{V}_h. \tag{5}$$

Remark 7. The equivalence of the HOMFLY polynomial and the generating function of Γ-semimodules $I(\Gamma;q)$ in (4) was pointed out by Chavan ([2]).

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Balayage in minimum Riesz energy problems with external fields

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This talk deals with a minimum energy problem in the presence of external fields on $\mathbb{R}^n$, $n \geqslant 2$, the energy being evaluated with respect to the α-Riesz kernel $\kappa_\alpha(x,y) := |x-y|^{\alpha-n}$, where $\alpha \in (0,n)$ and $\alpha \leqslant 2$. (Here $|x-y|$ is the Euclidean distance between $x, y \in \mathbb{R}^n$.) For precise formulations, we denote by $\mathfrak{M}$ the linear space of all (real-valued Radon) measures μ on $\mathbb{R}^n$, equipped with the *vague* topology of pointwise convergence on the continuous functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ of compact support, and by $\mathfrak{M}^+$ the cone of all positive $\mu \in \mathfrak{M}$. Given $\mu, \nu \in \mathfrak{M}$, we define the *mutual energy* and *potential* by means of

$$I(\mu,\nu) := \int \kappa_\alpha(x,y)\, d(\mu\otimes\nu)(x,y) \quad \text{and} \quad U^\mu(x) := \int \kappa_\alpha(x,y)\, d\mu(y), \quad x \in \mathbb{R}^n,$$

respectively, provided the value on the right is well defined as a finite number of $\pm\infty$. For $\mu = \nu$, $I(\mu,\nu)$ defines the *energy* $I(\mu) := I(\mu,\mu)$. A crucial fact is that κ_α is *strictly positive definite* in the sense that for any $\mu \in \mathfrak{M}$, $I(\mu)$ is $\geqslant 0$ whenever defined, and moreover $I(\mu) = 0 \iff \mu = 0$. This implies that all $\mu \in \mathfrak{M}$ with $I(\mu) < \infty$ form a pre-Hilbert space $\mathcal{E}$ with the inner product $\langle \mu, \nu\rangle := I(\mu,\nu)$ and the norm $\|\mu\| := \sqrt{I(\mu)}$. The topology on $\mathcal{E}$ defined by $\|\cdot\|$ is said to be *strong*. Moreover, κ_α is *perfect*, which means that the cone $\mathcal{E}^+ := \mathcal{E} \cap \mathfrak{M}^+$ is *strongly complete*, while the strong topology on $\mathcal{E}^+$ is *finer* than the induced vague topology on $\mathcal{E}^+$. (See Landkof's book [3] and historical notes therein.)

Fixing $A \subsetneqq \mathbb{R}^n$, we denote by $\mathcal{E}^+(A)$ the class of all $\mu \in \mathcal{E}^+$ *concentrated on* A, which means that $A^c := \mathbb{R}^n \setminus A$ is μ-negligible. (For closed A, $\mathcal{E}^+(A)$ consists of all $\mu \in \mathcal{E}^+$ with support $S(\mu) \subset A$.) Also fix an *external field* $f := -U^\vartheta$, where $\vartheta \in \mathfrak{M}^+$ is given. The problem in question is that on minimizing the *Gauss functional* $I_f(\mu)$, which sometimes is also referred to as the *f-weighted energy*, where

$$I_f(\mu) := \|\mu\|^2 + 2\int f\, d\mu = \|\mu\|^2 - 2I(\mu,\vartheta)$$

and μ ranges over $\mathcal{E}^1(A) := \big\{\mu \in \mathcal{E}^+(A) : \ \mu(\mathbb{R}^n) = 1\big\}$. That is, *does there exist* $\lambda_{A,f} \in \mathcal{E}^1(A)$ *with*

$$I_f(\lambda_{A,f}) = \inf_{\mu\in\mathcal{E}^1(A)} I_f(\mu)? \tag{1}$$

The investigation of this problem, initiated by Gauss, is still of interest due to its important applications in various areas of mathematics (see e.g. Saff and Totik [5] and numerous references therein).

If $A := K$ is compact while $f|_K$ is finitely continuous, then $\lambda_{K,f}$ does exist, for $I_f(\cdot)$ is vaguely lower semicontinuous, whereas $\mathcal{E}^1(K)$ is vaguely compact [1, Section III.1, Corollary 3 to Proposition 15]. However, these arguments, based on the vague topology only, fail down if A is noncompact, and the problem becomes "rather difficult" (Ohtsuka [4, p. 219]). To examine problem (1) for non-compact A, we developed an approach based on the perfectness of κ_α, whence on *both the strong and vague topologies* on $\mathcal{E}$ (see [10, 11]). To this end, we need to impose on A and ϑ the following three requirements:

• *The cone $\mathcal{E}^+(A)$ is strongly closed, whence strongly complete.* (As shown in [10, Theorem 3.9], this in particular holds if A is closed or even *quasiclosed*. By Fuglede [2], the latter means that A can be approximated in outer capacity by closed sets. For the concepts of *outer* and *inner capacities*, see e.g. Landkof [3, Section II.2.6]. It is worth noting here that a quasiclosed set is *not* necessarily Borel.)

• *$c_*(A) > 0$, where $c_*(\cdot)$ stands for the inner capacity of a set; or equivalently $\mathcal{E}^1(A) \neq \varnothing$.*

• *$\vartheta \in \mathfrak{M}^+$ is bounded, i.e. $\vartheta(\mathbb{R}^n) < \infty$, and moreover*

$$\inf_{(x,y)\in S(\vartheta)\times A} |x-y| > 0.$$

Then, the *inner κ_α-balayage* ϑ^A of ϑ to A can be defined as the unique bounded measure in $\mathcal{E}^+(A)$ such that $U^{\vartheta^A} = U^\vartheta$ n.e. on A, i.e. on all of A except for a set with $c_*(\cdot) = 0$; see [10, Theorem 4.7(iii$_1$)]. (For the general theory of inner κ_α-balayage, we refer to [6, 7], cf. also [8, 9].) This implies that $I_f(\cdot)$ is strongly continuous on $\mathcal{E}^+(A)$, which is crucial to the analysis of problem (1), performed in [10, 11].

Theorem 1 (see [11, Theorem 2.6])**.** *For $\lambda_{A,f}$ to exist, it is necessary and sufficient that*

$$c_*(A) < \infty \quad \text{or} \quad \vartheta^A(\mathbb{R}^n) \geqslant 1. \tag{2}$$

By [7, Definition 2.1], $Q \subset \mathbb{R}^n$ is said to be *not inner α-thin at infinity* if

$$\sum_{j\in\mathbb{N}} \frac{c_*(Q_j)}{q^{j(n-\alpha)}} = \infty,$$

where $q \in (1,\infty)$ and $Q_j := Q \cap \{y \in \mathbb{R}^n : \ q^j < |y| \leqslant q^{j+1}\}$. The inner κ_α-balayage of any $\mu \in \mathfrak{M}^+$ to such Q preserves its total mass [7, Corollary 5.3], whence the following corollary to Theorem 1 holds.

Corollary 2. *If A is not inner α-thin at infinity, then $\lambda_{A,f}$ exists if and only if $\vartheta(\mathbb{R}^n) \geqslant 1$.*

Theorem 3 (see [11, Theorem 2.10])**.** *Assume* (2) *is fulfilled, and moreover $\vartheta^A(\mathbb{R}^n) \leqslant 1$. Then*

$$\lambda_{A,f} = \begin{cases} \vartheta^A + c_{A,f}\gamma_A & \text{if } \ c_*(A) < \infty, \\ \vartheta^A & \text{otherwise,} \end{cases} \tag{3}$$

where $c_{A,f} \in [0,\infty)$, while γ_A is the inner κ_α-equilibrium measure on A, normalized by $\gamma_A(\mathbb{R}^n) = c_*(A)$.

For the inner κ_α-equilibrium measure on the set A in question, see [9, Theorem 7.2] with $\kappa := \kappa_\alpha$.

In the following Theorems 4 and 5, A is assumed to be *closed*. The *reduced kernel* $\breve{A}$ of A is the set of all $x \in A$ such that $c_*(A \cap U_x) > 0$ for any neighborhood U_x of x in $\mathbb{R}^n$, cf. [3, p. 164].

Theorem 4 (see [11, Theorem 2.11])**.** *Under the requirements of Theorem 3, assume moreover that A^c is connected unless $\alpha < 2$. Then, by virtue of the representation* (3) *and* [6, Theorems 7.2, 8.5],

$$S(\lambda_{A,f}) = \begin{cases} \breve{A} & \text{if } \ \alpha < 2, \\ \partial_{\mathbb{R}^n}\breve{A} & \text{otherwise.} \end{cases}$$

Theorem 5 (see [10, Theorem 2.22])**.** *If A is not α-thin at infinity and $\delta(\mathbb{R}^n) > 1$, then $S(\lambda_{A,f})$ is compactly supported in A.* (Compare with Theorem 4. Note that $\lambda_{A,f}$ does exist, see Corollary 2.)

Theorems 4, 5 give an answer to the question raised by Ohtsuka in [4, p. 284, Open question 2.1].

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Table of contents